

UNIVERSITY OF CALIFORNIA SAN DIEGO

Geometry and Mechanics of Fluid Surfaces

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by

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University of California San Diego

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PREFACE

How does matter organize into form? How do flows, shapes, and structures emerge and evolve? These questions lie at the intersection of physics, biology, mathematics, philosophy, and engineering, and motivate the central theme of this thesis.

Placed in a broader context, this thesis can be understood as a projection of geometry onto these interconnected fields. In biology, geometry is tied to form, function, and morphogenetic self-organization. In mechanics, geometry and topology fundamentally shape the behavior of fluids and elastic bodies. In mathematics, differential geometry provides an invariant language for describing continua independent of coordinates. At a structural level, geometric and variational principles point toward a unifying natural philosophy of form and motion.

This thesis develops geometric and variational formulations, together with computational methods, for fluid and elastic surfaces with complex geometry and topology commonly found in biological systems. The broader goal is to establish a common language linking geometry, mechanics, computation, and living systems, based on the view that geometry is not merely a description of form, but a physical principle governing its emergence and evolution.

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Chapter 1, in full, is a reprint of the material as it appears in “Stokes Flow of an Evolving Fluid Film with Arbitrary Shape and Topology,” co-authored with David Saintillan and Albert Chern. The dissertation author was the primary investigator and author of this paper.

Chapter 2, in full, is a reprint of the material as it appears in “Viscous Vortex Dynamics on Surfaces,” co-authored with Hang Yin and Albert Chern. The dissertation author and Hang Yin were co-primary investigators and co-lead authors of this work.

Chapter 3, in full, is a reprint of the material as it appears in “Active Nematic Fluids on Riemannian 2-Manifolds,” co-authored with David Saintillan and Albert Chern. The dissertation author was the primary investigator and author of this paper.

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ABSTRACT OF THE DISSERTATION

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Fluid dynamics on curved and evolving surfaces arises throughout soft matter physics, biology, and morphogenesis, where geometry and topology fundamentally shape flow behavior, yet classical fluid theories remain largely restricted to simple stationary geometries. This thesis develops a unified geometric and variational framework for fluid mechanics on arbitrary surfaces using coordinate-free differential geometry, Onsager’s variational principle, and structure-preserving discretization. We first reformulate the evolving surface Stokes equations into an invariant variational form that avoids tangent–normal decomposition and naturally yields stable numerical schemes

on surfaces of arbitrary shape and topology. We then derive a vorticity-based formulation of the Navier–Stokes equations on curved surfaces with nontrivial topology, orientability, and boundary curvature, revealing curvature-dependent effects in viscous forces, vortex dynamics, harmonic modes, and vorticity generation. Finally, we extend the framework to active nematic fluids by representing orientational order as sections of a complex line bundle transported by the Lie derivative, providing a natural treatment of defects, curvature, and advection on general manifolds. Together, these results establish geometric and variational principles as a coherent foundation for modeling and computing fluid flows on curved and evolving surfaces, while highlighting geometry and topology as active drivers of soft matter and biological dynamics.

Introduction

Fluid dynamics on curved and evolving surfaces arises naturally in a wide range of physical and biological systems across many scales. When surface dissipation dominates bulk dissipation, viscous flows of evolving fluid films govern phenomena as diverse as foam stability, lipid membrane biophysics, and epithelial tissue dynamics. In these settings, the underlying substrate is not a fixed Euclidean domain but a deformable manifold whose geometry and topology evolve in time. Similarly, active nematic systems—defined by intrinsic activity coupled to orientational order—exhibit behavior fundamentally shaped by confinement on curved surfaces, with applications ranging from collective dynamics in motile bacterial suspensions, cytoskeletal organization, and tissue morphogenesis to engineering of soft materials. Even classical Navier-Stokes vortex dynamics becomes qualitatively altered when restricted to non-Euclidean geometries, where viscous response give rise to geometry-induced vorticity production.

This geometric complexity fundamentally shapes flow behavior, influencing transport, stress generation, and pattern formation. Yet classical formulations of fluid mechanics are largely built on flat domains and coordinate-based descriptions, making their extension to general surfaces both conceptually and computationally challenging. More specifically, a classical incompressible Newtonian fluid is governed by the celebrated Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{Re} \Delta \mathbf{u} - \nabla p + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0. \quad (1)$$

On curved manifolds, familiar notions such as gradients, divergence must be reformulated using differential geometry. The tangential velocity in surface fluids becomes a tensor field on the manifold, requiring specialized techniques for covariant differentiation and complicating numerical techniques, and the original equations is subjects to nontrivial coupling with surface geometry. When surfaces evolves, fluid also moves in the normal direction and deforms the domain. The corresponding governing equations often rely on explicit tangent-normal decompositions. In Scriven's evolving Stokes equations [108, 6], the fully viscous limit ($Re \rightarrow 0$) of Equation 1 generalized to curved and evolving surface, we have

$$\begin{cases} (\Delta + K)\mathbf{u} + 2Q(\text{grad } u_n) + 2u_n \text{grad } H - \text{grad } p + \mathbf{f} = 0, \\ -2 [\langle \nabla \mathbf{u}, \mathbb{I} \rangle + u_n(4H^2 - 2K)] + 2pH + f_n = 0, \\ \text{div } \mathbf{u} + 2Hu_n = 0. \end{cases} \quad (2)$$

This classical tangent-normal formulation involves the tangent velocity $\mathbf{u}$ and normal velocity u_n , subject to covariant differentiation ∇ and intimately coupled to the surface curvatures $H, K, \mathbb{I}$, and Q . While the governing equations date back to 1960, their practical use has remained limited due to their complexity in general geometries. Classical approaches based on local coordinates often lead to large systems of equations, rendering the formulations cumbersome, difficult to solve in practice, and frequently restricted to specific classes of geometries.

On surfaces, one convenient representation of the Navier–Stokes equations is through the vorticity formulation. The vector-valued Equation 1 reduces to a scalar transport equation for the vorticity, thereby avoiding many of the complications associated with covariant differentiation. However, on curved surfaces, vorticity dynamics is fundamentally altered by geometry in ways that have been historically overlooked. A critical overlooked detail is that the *Gaussian curvature* K of the surface contributes an

additional term to the viscous force. In the vorticity formulation, this manifests as:

$$\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = \frac{1}{Re}(\Delta w + 2 \operatorname{curl}(K\mathbf{u})), \quad (3)$$

where w is scalar vorticity being transported on surface. Thus, viscosity not only diffuses vorticity but also produce it, in a curvature dependent manor. Geometric contributions also affect the evolution of harmonic components of the velocity field on non-simply-connected domains [127]. These geometric terms are essential for capturing physically accurate behavior of fluids on curved surfaces.

Similarly challenges arise in complex fluid, such as active nematic systems, where the dynamics involves the evolution of fluid's orientational anisotropy. These systems—exemplified by collective dynamics in motile bacterial suspensions, the structural organization of cytoskeletal assemblies, and morphogenetic processes in developing tissues—exhibit behavior fundamentally shaped by the interplay between activity, geometry, and topology. Recent experimental discoveries have sparked growing interest in understanding how active nematic dynamics depends on geometric and topological constraints imposed by spatial confinement. Modeling such systems on curved surfaces presents new challenges: the order parameter must be transported consistently with the manifold's intrinsic geometry, and the coupling between curvature, topology, and nematic defects introduces rich mathematical structure. Traditional approaches often rely on explicit local coordinates, which obscure geometric structure, couple equations unnecessarily, and obstruct structure-preserving discretizations. These challenges are amplified in modern applications, where one must simultaneously account for evolving geometry, internal order, and complex boundary or topological effects.

This thesis develops a unified geometric and variational framework for fluid mechanics on arbitrary curved and evolving surfaces. The central philosophy is to formulate physical laws in a coordinate-free, differential-geometric language, and to derive

governing equations through variational principles. This perspective leads to compact, intrinsic formulations that naturally encode geometric effects, directly translates into structure-preserving discrete models defined on triangulated meshes, unifies diverse physical phenomena under a single framework, and reveals new geometric insights that would be obscured in coordinate-dependent formulations.

We begin by revisiting the classical theory of viscous flow on deformable interfaces. In Chapter 1, we reformulate Equation (2) using Onsager’s variational principle, deriving a coordinate-free representation of evolving surface Stokes flow that avoids explicit tangent-normal splitting. This formulation clarifies the underlying mechanics and leads directly to a discrete variational integrator defined on triangulated meshes, enabling stable computation on surfaces of arbitrary shape and topology. In the same chapter, we complete the continuum modeling of deformable viscous fluid membrane, such as lipid membranes and epithelial tissue layers, which exhibit solid-like behavior in the normal direction. This is achieved by coupling the evolving Stokes equations with the Helfrich bending model. The resulting Helfrich bending forces are derived variationally within the framework of discrete differential geometry and admit a direct correspondence with the governing partial differential equations in the smooth continuum setting.

In Chapter 2, we derive a complete vorticity-based formulation of the Navier–Stokes equations on general surfaces, including Equation 3. The formulation applies to surfaces with nontrivial topology, non-orientable structure, and boundaries possessing intrinsic curvature. This work builds upon prior results on the vortex dynamics of the Euler equations [127], extending their topological framework to incorporate the additional effects arising from viscosity. We show that these viscous geometric contributions are essential for correctly capturing fluid behavior, including the conservation of rigid-body motions. In addition, we prove that curvature-sheet singularities can, through viscous response, generate jump discontinuities in the vorticity field relative to the fluid velocity.

This result leads to several advances in the understanding of geometric effects in fluid dynamics, as well as new computational approaches that remain relevant even for flows in flat Euclidean space. A central observation is that Gaussian curvature and boundary geodesic curvature naturally combine into a unified Gauss–Bonnet curvature density governing vorticity evolution. Within this geometric framework, the nondissipative natural boundary condition for velocity—the free-slip condition—reduces to a simple homogeneous Dirichlet condition for the vorticity, even in the presence of curved boundaries. Frictional boundary effects can then be interpreted geometrically as modifications of the boundary geodesic curvature toward increasingly negative curvature, with the no-slip condition emerging as the limiting case of an infinitely hyperbolic boundary. Moreover, the Kutta condition at sharp aerodynamic interfaces arises naturally from frictionless free-slip boundaries, without requiring explicit transitional boundary-layer models. The resulting formulation provides new geometric interpretations of classical fluid phenomena while producing efficient numerical schemes on arbitrary geometries.

In Chapter 3, we develop a geometric theory of active nematic fluids on Riemannian surfaces. In the existing literature, the prevailing framework for modeling nematics is the Landau–de Gennes Q -tensor theory, in which the dynamics are formulated in terms of traceless symmetric tensors. In two dimensions, however, an equivalent and more geometrically natural description exists in terms of complex scalar fields, which furthermore extends naturally to general k -atic order parameters. On curved surfaces, we represent both the velocity field and the nematic order parameter as sections of complex line bundles. The elasticity of k -atic fields can then be formulated using covariant derivatives on these bundles, while their transport is described through Lie derivatives. This approach naturally incorporates curvature, topology, and topological defects through the underlying bundle structure, and unifies the treatment of fluid flow and nematic advection within a single geometric language. Moreover, the formulation leads to a gauge-invariant discretization and efficient numerical schemes for simulating

nematic dynamics on arbitrary surfaces. It also reveals how geometric structures—such as connections, curvature, and bundle topology—govern the interplay between flow and internal orientational order.

Taken together, the results of this thesis demonstrate that geometry and mechanics are inseparable in the study of fluid surfaces. A coordinate-free, variational formulation provides a unifying language that clarifies the structure of governing equations across different flow regimes, naturally incorporates curvature and topology as organizing principles, and enables structure-preserving numerical methods on arbitrary geometries. Across viscous surface flows, active matter systems, and inertial vortex dynamics, geometry emerges not as a complication but as a fundamental organizing principle that guides flow behavior. By bridging differential geometry, continuum mechanics, and computation, this work provides a flexible and unified framework for studying complex fluid systems on curved and evolving domains, with applications ranging from soft matter physics and engineering to biological morphogenesis and tissue mechanics.

Chapter 1

Stokes flow of an evolving surface

This chapter establishes the first part of the thesis by reformulating the evolving surface Stokes equations (Equations 2), originally formulated by Scriven in 1960, and developing a foundational geometric and variational framework for fluid mechanics on curved and evolving surfaces. The resulting framework and numerical methods provide the geometric language that Chapters 2 and 3 extend to inertial vortex dynamics and active nematic fluids, respectively.

This chapter is based on the paper *Stokes Flow of an Evolving Fluid Film with Arbitrary Shape and Topology* by Cuncheng Zhu, David Saintillan, and Albert Chern. The dissertation author was the primary investigator and lead author of this work.

1.1 Introduction

When surface dissipation dominates bulk dissipation (i.e., when the Saffman-Delbrück length far exceeds the system size), the viscous flow of evolving fluid films can be modeled without accounting for the bulk fluid [6, 100]. The governing equations, known as the evolving Stokes equations, have not only been of theoretical interest, but also found practical applications in engineering, such as the study of foam [39]. Their use also has a long-standing history in biophysical contexts, addressing fundamental problems across scales—from modeling subcellular structures like lipid bilayers to

tissue-level phenomena such as epithelial monolayers [1].

Historically, the governing equations were formulated by Scriven [108] to study foam instability. When coupled with bending elasticity, the evolving Stokes equations serve as a common model for lipid membranes [6]. Recently, there has been substantial interest in modeling cell and tissue growth by coupling the viscous layer with additional anisotropy and active processes, such as those seen in the cellular cortex and active nematic fluids [2, 67, 117]. The evolving Stokes equations, along with extensions to the full Navier-Stokes equations, have been independently derived through various principles [108, 6, 117, 61, 97, 102, 71, 2, 98] and are compared in Brandner et al. [16]. Our approach is closest to the formulation of Arroyo & DeSimone [6] via the Onsager variational principle for dissipative systems.

To computationally solve the evolving Stokes equations, most existing methods tackle the partial differential equations by explicitly splitting them into normal and tangential components. This involves a vector-valued equation for the tangent velocity and a scalar equation for the normal velocity on a manifold. When viewed this way, the system of equations is challenging to solve. First, the tangent equation is tensor-valued on an evolving Riemannian manifold, necessitating specialized techniques for covariant differentiation [80, 47, 60, 116, 123]. Some methods avoid the tensor-valued equation of tangent velocity through streamfunction formulations, although this approach is limited to simply-connected domains [117] unless additional topological techniques are employed [127]. Second, the equations in the decomposed form are explicitly coupled with surface curvatures, which, in classic numerical approximations, require representations using high-order basis functions [102, 63].

In this study, instead of directly tackling the classic equations, we return to the fundamental governing kinematics and principles of a viscous surface film. We replace the system of equations expressed in tangent-normal splitting with a simple, coordinate-free differential-geometric formulation. We show that our resulting equations agree with

earlier formulations by Scriven [108] and Arroyo & DeSimone [6] but in a more elegant form. Our approach also directly carries over to the discrete setting. The abstraction directly leads to a discretization for the strain rate tensor on discrete meshes. We then construct a discrete Rayleighian for the system and derive the discrete Stokes equations via the Onsager variational principle, leading to a simple linear system. We provide a self-contained exposition of both the continuous and discrete models, as well as the numerical methods used to solve the system of equations on arbitrary geometries and topologies.

1.2 Theory

We derive the equations of motion for an evolving surface driven by viscous Stokes flow using Onsager’s variational principle. Specifically, a dissipation functional, called the Rayleighian, is assigned to each position and velocity state of a deformable surface. The equations of motion are then obtained by finding the velocity that minimizes this functional.

We adopt a differential geometric approach, similar to Marsden & Hughes [66], to describe continuum mechanics. Tensors are expressed using fiber bundle notation for clarity. Readers unfamiliar with this terminology can refer to §1.2.1 for a brief overview. We first define relevant tensors and differential operators in §1.2.2, followed by the definition of the Rayleighian for the system in §1.2.3. For simplicity in presenting our smooth theory, all functions and tensors are assumed to be of C^∞ class.

1.2.1 Terminology and notation

On a manifold M , the *tangent space* $T_p M$ is a vector space that provides a local linear approximation of the manifold at $p \in M$. The disjoint union of all tangent spaces forms the *tangent bundle* $TM = \bigcup_p T_p M$, where each $T_p M$ is also called the *fiber* of the tangent bundle TM at the *base point* $p \in M$ on its *base manifold* M . The dual bundle

of TM is the *cotangent bundle*, T^*M , whose fibers are the dual spaces of the tangent spaces. The (m, n) -typed tensor bundle is written as $TM^{\otimes m, n} = TM^{\otimes m} \otimes T^*M^{\otimes n}$. A tensor field Ψ on M is a *section* of the tensor bundle, denoted as $\Psi \in \Gamma(TM^{\otimes m, n})$, which assigns a tensor $\Psi|_p \in T_p M^{\otimes m, n}$ to each point $p \in M$. When a basis section is chosen, its components, $\Psi_{i_1 \dots i_n}^{j_1 \dots j_m}$, have m contravariant indices and n covariant indices.

For a map $\varphi : M \rightarrow W$, where W is another manifold, the *pullback bundle* $T_\varphi W = \varphi^*TW$ is defined with M as its base manifold, while the fiber at $p \in M$, $(T_\varphi W)|_p = T_{\varphi(p)}W$, is the tangent space to W at $\varphi(p) \in W$.

Note that tensors can be identified with linear maps; for example, $\Psi|_p \in T_p M^{\otimes m, n}$ acts as a linear map $\Psi|_p : T_p M \rightarrow T_p M^{\otimes m, n-1}$.

1.2.2 Differential geometry of an evolving surface

We describe a deformable surface as follows. Let M be a 2-dimensional closed manifold ($\partial M = \emptyset$) with arbitrary genus, representing the *material/Lagrangian space*. Let W be a 3-dimensional Riemannian manifold representing the *world/Eulerian space*. Let $g \in \Gamma(T^*W \otimes_{\text{symm}} T^*W)$ (i.e., a symmetric $(0, 2)$ tensor field) denote the Riemannian metric tensor on W . The common setup is to set $W = \mathbb{R}^3$, the Euclidean 3-space, with $g[V, W] = V \cdot W$ being the Euclidean inner product for any $V, W \in \mathbb{R}^3$. However, our theory is not limited to this assumption. The *position* of a deformable surface is an embedding function $\varphi : M \hookrightarrow W$ from the material space to the world space. A time-dependent deformable surface, or an evolving surface, is a time-dependent embedding $\varphi(t) = \varphi_t : M \hookrightarrow W$. The *material velocity* of an evolving surface $\varphi(t)$ is given by $\dot{\varphi}_t := \partial_t \varphi_t$, which is a *pullback* vector field $\dot{\varphi}_t \in \Gamma(T_{\varphi_t} W)$ with its base point in the material frame M (cf. §1.2.1).¹

Each material velocity $\dot{\varphi}_t \in \Gamma(T_{\varphi_t} W)$ can be extended to a *spatial velocity* over

¹The velocity $\dot{\varphi}_t|_p$ is evaluated at each point $p \in M$ in the material domain, and its value $\dot{\varphi}_t|_p \in T_{\varphi_t(p)}W$ is a tangent vector to the world space based at $\varphi_t(p) \in W$.

W , denoted by $\mathbf{U}_t \in \Gamma(TW)$, such that $\dot{\varphi}_t = \mathbf{U}_t \circ \varphi_t$, or equivalently, $\dot{\varphi}_t|_p = \mathbf{U}_t|_{\varphi_t(p)}$ for $\varphi_t(p) \in W$. Note that this extension is not unique, as the assignment of $\mathbf{U}_t$ at locations away from the surface $\varphi_t(M) \subset W$ is arbitrary.

Deformation gradient and Cauchy–Green tensor

The differential of the positioning $\varphi: M \hookrightarrow W$ of a surface is denoted by $F := d\varphi \in \Gamma(T_\varphi W \otimes T^*M)$. This differential is also known as the pushforward map, the tangent map, or the *deformation gradient*. At each point $p \in M$, the two-point tensor $F|_p$ can be identified with a linear map (cf. §1.2.1) $F|_p = d\varphi|_p: T_pM \rightarrow T_{\varphi(p)}W$ ² that realizes a material tangent vector as a 3D world vector, as expected for the deformation gradient.

The embedding $\varphi: M \hookrightarrow W$ induces a Riemannian metric (the first fundamental form) $I \in \Gamma(T^*M \otimes_{\text{symm}} T^*M)$ on M from the Riemannian structure g on W . This induced metric is defined by $I[v, w] := g[Fv, Fw] = v^\top F^\top g F w$ for each $v, w \in T_pM$, $p \in M$. In continuum mechanics, $I = F^\top g F$ is known as the *right Cauchy–Green tensor*.³

For brevity, we denote the induced metric as $\langle \cdot, \cdot \rangle = I[\cdot, \cdot]$ and the L_2 inner product as $\langle\langle \cdot, \cdot \rangle\rangle = \int_M \langle \cdot, \cdot \rangle dA$ where dA is the area form from the metric I . These inner products define the vector norm $|\cdot|^2$ and the L_2 norm $\|\cdot\|^2$. Dual pairings between primal (contravariant) and dual (covariant) tensors are denoted by $\langle \cdot | \cdot \rangle$ and $\langle\langle \cdot | \cdot \rangle\rangle = \int_M \langle \cdot | \cdot \rangle dA$. The L_2 inner product between two scalar functions f and g is denoted by $\langle\langle f, g \rangle\rangle = \int_M f g dA$.

²At each point $p \in M$, the tensor $F|_p \in T_{\varphi(p)}W \otimes T_p^*M$ pairs with a vector in T_pM and returns a value in $T_{\varphi(p)}W$. Since it relates quantities in two different spaces M and W , it is called a two-point tensor.

³Here we identify tensors I, g, F as linear maps $I_p: T_pM \rightarrow T_p^*M$, $g_q: T_qW \rightarrow T_q^*W$, $F_p: T_pM \rightarrow T_{\varphi(p)}W$. Then we have the compositional relation $I = F^\top g F$, where $F_p^\top: T_{\varphi(p)}^*W \rightarrow T_p^*M$ is the adjoint of F_p .

Strain rate tensor

For an evolving surface φ_t , the induced metric $I_t \in \Gamma(T^*M \otimes_{\text{symm}} T^*M)$ varies with time t . The *strain rate tensor* $E_t \in \Gamma(T^*M \otimes_{\text{symm}} T^*M)$ is defined as half the rate of change of the induced metric: $2E_t := \dot{I}_t$.

To compute the time derivative of $I_t = F_t^\top g F_t$ we first calculate the time derivative of F_t . Let $\mathbf{U}_t \in \Gamma(TW)$ be any extended velocity field such that $\dot{\varphi}_t = \mathbf{U}_t \circ \varphi_t$. Then,

$$\dot{F} = d\dot{\varphi}_t = d(\mathbf{U}_t \circ \varphi_t) = \nabla(\mathbf{U}_t \circ \varphi_t) \circ d\varphi_t \equiv (\nabla \mathbf{U}_t)F, \quad (1.1)$$

where $\nabla : \Gamma(TW) \rightarrow \Gamma(TW \otimes T^*W)$ is the Levi-Civita covariant derivative compatible with the metric g . When applied to a vector $v \in \Gamma(TM)$, $\dot{F}_t v = \nabla_{F_t v} \mathbf{U}_t$, which is a directional derivative of $\mathbf{U}_t$ tangential to the surface. In particular, (1.1) depends only on the values of $\mathbf{U}_t \circ \varphi_t$ at the surface and not on the choice of its extension $\mathbf{U}_t$ over W .

With a time-invariant metric g and (1.1), the strain rate tensor is given by

$$2E_t = \dot{I}_t = \dot{F}_t^\top g F_t + F_t^\top g \dot{F}_t = F_t^\top ((\nabla \mathbf{U}_t)^\top g + g \nabla \mathbf{U}_t) F_t. \quad (1.2)$$

When applied to vectors, $2E_t \llbracket v, w \rrbracket_p = (\nabla_{F_t v} \mathbf{U}_t) \cdot (F_t w) + (\nabla_{F_t w} \mathbf{U}_t) \cdot (F_t v)$ for $v, w \in \Gamma(T_p M)$. Again, E_t depends only on the values of $\mathbf{U}_t \circ \varphi_t$ and not on the choice of its extension $\mathbf{U}_t$.

Differential operators

The operator that takes each velocity field $\dot{\varphi} = \mathbf{U} \circ \varphi$ at the surface to its induced strain rate tensor (1.2) is called the Killing operator:

$$\mathcal{K} : \Gamma(T_\varphi W) \rightarrow \Gamma(T^*M \otimes_{\text{symm}} T^*M), \quad \mathcal{K}\dot{\varphi} := E = F^\top \frac{(\nabla \mathbf{U})^\top g + g \nabla \mathbf{U}}{2} F. \quad (1.3)$$

The trace of the strain rate tensor defines the *divergence operator*

$$\text{DIV} : \Gamma(T_\phi W) \rightarrow C^\infty(M), \quad \text{DIV } \dot{\phi} := \text{tr}(\mathcal{K} \dot{\phi}), \quad (1.4)$$

which measures the rate of change of area induced by the given velocity field. Define the *gradient operator* as the negative adjoint of the divergence operator:

$$\text{GRAD} = -\text{DIV}^* : C^\infty(M) \rightarrow \Gamma(T_\phi^* W), \quad \langle\langle \text{GRAD } f | \dot{\phi} \rangle\rangle = -\langle\langle f, \text{DIV } \dot{\phi} \rangle\rangle \quad (1.5)$$

for all $f \in C^\infty(M)$. The *covariant divergence* is the negative adjoint of the Killing operator:

$$\text{DIV}^\nabla = -\mathcal{K}^* : \Gamma(TM \otimes_{\text{symm}} TM) \rightarrow \Gamma(T_\phi^* W), \quad \langle\langle \text{DIV}^\nabla T | \dot{\phi} \rangle\rangle = -\langle\langle T | \mathcal{K} \dot{\phi} \rangle\rangle \quad (1.6)$$

for all symmetric tensor fields $T \in \Gamma(TM \otimes_{\text{symm}} TM)$. Here the dual pairing $\langle T | E \rangle$ between $(0,2)$ and $(2,0)$ tensors is $\langle T | E \rangle = \text{tr}(T^\top E) = \sum_{ij} T^{ij} E_{ij}$.

Notably, the operators $\mathcal{K}$, DIV , GRAD , and DIV^∇ , derived from the strain rate tensor E of the evolving surface, differ from the commonly used intrinsic operators $\mathcal{K}$, div , grad , and div^∇ . As shown through the normal-tangent decomposition in Appendix A, these differences arise from additional contributions related to curvature (cf. 1.15–1.17).

1.2.3 Onsager's variational principle for an evolving viscous fluid film

The relaxation dynamics of an evolving viscous fluid film is governed by the Onsager's variational principle, which states that Stokes flow follows the dynamics of the least energy dissipation [33, 6]. The dissipation functional, or Rayleighian, captures the system's dissipation rate, and the governing flow is the stationary condition of this functional subject to the incompressibility constraint.

Consider a viscous dynamical system where the dissipation functional $\mathcal{D}$ comprises the rate of viscous dissipation, and power inputs from an external force $\mathbf{B} \in \Gamma(T^*W)$ and a shape-dependent potential energy V_φ :

$$\begin{aligned}\mathcal{D}(\mathbf{U}) &= \frac{1}{2} \langle\langle \mathbf{E} | \mathbf{T} \rangle\rangle - \langle\langle \mathbf{B} | \mathbf{U} \rangle\rangle + \dot{V}_\varphi \\ &= \frac{1}{2} \int_M \langle \mathbf{E} | \mathbf{T} \rangle \, dA - \int_{\varphi(M)} \langle \mathbf{B} | \mathbf{U} \rangle \, dA + \int_M \langle \delta_\varphi V_\varphi | \dot{\varphi} \rangle \, dA.\end{aligned}\tag{1.7}$$

Here, $\mathbf{T} = 2\mu\mathbf{E} \in \Gamma(TM \otimes_{\text{symm}} TM)$ is the viscous stress tensor (assumed Newtonian with constant viscosity μ), $\delta_\varphi V_\varphi$ is the conservative force obtained by taking the variation of V_φ , and $\dot{\varphi} = \mathbf{U} \circ \varphi$ is the material velocity.

Onsager's variational principle determines the Stokes flow by solving the constrained optimization problem:

$$\min_{\mathbf{U}} \mathcal{D}(\mathbf{U}) \quad \text{subject to} \quad \text{DIV } \mathbf{U} = 0.\tag{1.8}$$

Using the fluid pressure $p \in C^\infty(M)$ as a Lagrange multiplier, this problem can be formulated as a minimax problem involving the augmented Rayleighian $\mathcal{R}$:

$$\min_{\mathbf{U}} \max_p \mathcal{R}(\mathbf{U}, p), \quad \text{where} \quad \mathcal{R}(\mathbf{U}, p) = \mathcal{D}(\mathbf{U}) + \langle\langle p, \text{DIV } \mathbf{U} \rangle\rangle.\tag{1.9}$$

On a closed surface, the stationary conditions yield the Stokes equations for an evolving surface:

$$2\mu \text{DIV}^\nabla \mathcal{K} \mathbf{U} - \text{GRAD } p + \mathbf{B} - \delta_\varphi V_\varphi = 0, \quad \text{DIV } \mathbf{U} = 0.\tag{1.10}$$

The symmetric, negative-definite, Laplacian-like operator $2 \text{DIV}^\nabla \mathcal{K} : \Gamma(TW) \rightarrow \Gamma(T^*W)$ quantifies the viscous force, and is thus termed viscosity Laplacian. Although not the main focus of this paper, a common free energy in this context is the Helfrich

Hamiltonian $V_\varphi = \int_W \kappa H^2 dA$, which results in the bending force $\mathbf{B}_\kappa = -\delta_\varphi V_\varphi = \kappa \mathbf{N}(\Delta H + 2H(H^2 - K))$, where H and K are the mean and Gaussian curvatures, respectively, κ is the bending modulus, and $\mathbf{N}$ is the surface normal [50]. This bending-driven relaxation dynamics is also used in the numerical examples of §1.4. The commonly used form of (1.10), expressed through normal-tangent decomposition, is derived in Appendix A.

Remarks on the evolving Stokes equations and their solvability

Note that the presence of a *Killing vector field* $\mathbf{X}$ ($\mathcal{K}\mathbf{X} = \mathbf{0}$) can make the solution to (1.10) non-unique. In that case we select the least-norm solution, effectively projecting out rigid motions. This treatment aligns with scenarios where the surface is immersed with friction in a bulk fluid. A simple way to model friction is by augmenting the Rayleighian to $\mathcal{R}^\alpha = \mathcal{R} + \alpha \|\mathbf{U} - \mathbf{U}_0\|^2/2$, where $\mathbf{U}_0$ is the bulk velocity and α is the friction coefficient. This introduces a friction force $\mathbf{B}_\alpha = \alpha(\mathbf{U} - \mathbf{U}_0)$ in the momentum equation. Such flow driven by the bulk fluid is used as a numerical example in §1.4.3. When $\mathbf{U}_0 = \mathbf{0}$, the solution approaches the least-norm solution as $\alpha \rightarrow 0$.

1.3 Methods

In this section, we present a simple numerical method to solve (1.10) on a triangle mesh with general geometry and topology. We begin by discretizing the strain rate tensor using finite elements, following the definition provided in (1.2). Based on the discretized strain rate, we construct the system's Rayleighian. A nonlinearly stable, structure-preserving variational integrator is derived based on the Onsager's variational principle. To keep the numerical scheme minimal, we demonstrate the method using first-order spatial and temporal discretizations.

We discretize the system on a closed triangular mesh M of arbitrary genus, consisting of vertices, edges, and faces $\{\mathfrak{V}, \mathfrak{E}, \mathfrak{F}\}$. The vertex positions are given by the

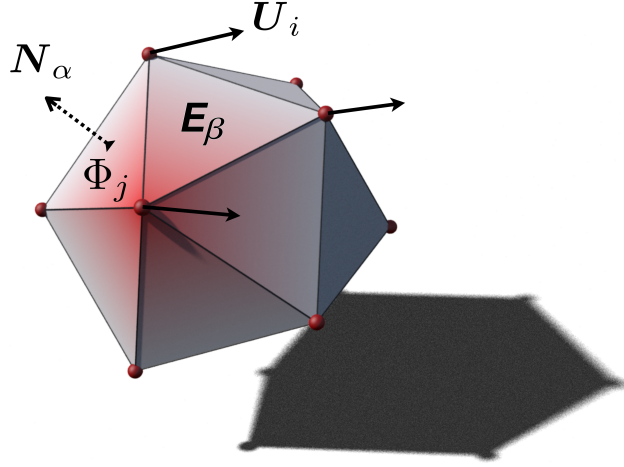


Figure 1.1. Discretization of the strain rate tensor. The velocity $\mathbf{U}_i$ is defined at each vertex i , with surface normals $\mathbf{N}_\alpha$ at each face α . The finite-element hat function Φ_j , shown by the color map, has local support around vertex j . Differentiating $\mathbf{U}_i$ via Φ_j produces $(\nabla \mathbf{U})_\beta$, which is symmetrized and projected using $\mathcal{P}_\beta = \mathbf{g} - \mathbf{N}_\beta \otimes \mathbf{N}_\beta$ to yield the strain rate tensor $\mathbf{E}_\beta$ on each face β .

realization $\varphi : M \hookrightarrow \mathbb{R}^3$. M has a tangent space $T_\alpha M$ and a surface normal $\mathbf{N}_\alpha$ at each face $\alpha \in \mathfrak{F}$ (cf. figure 1.1).

In the discrete setting, we express tensor-valued functions and operators using index notation, with upper indices denoting their components in Cartesian coordinates. Vertex and face elements are denoted with lower indices, where vertex indices are Roman letters and face indices are Greek letters. For example, we denote vertex positions as φ_j^i , face normals as $\mathbf{N}_\alpha^i$, face pressure as p_α , for $j \in \mathfrak{V}$, $\alpha \in \mathfrak{F}$, and $i \in \{1, 2, 3\}$.

1.3.1 Strain rate tensor and differential operators

To express the strain rate tensor $\mathbf{E}$ on M (cf. (1.2)) in Cartesian coordinate, we first realize it in $\mathbb{R}^3$. We denote this realization as $2\tilde{\mathbf{E}} = \mathcal{P}((\nabla \mathbf{U})^\top \mathbf{g} + \mathbf{g} \nabla \mathbf{U})\mathcal{P}$, which satisfies $\tilde{\mathbf{E}}[\mathbf{V}, \mathbf{W}] = \mathbf{E}[\mathcal{P}\mathbf{V}, \mathcal{P}\mathbf{W}]$ for $\mathbf{V}, \mathbf{W} \in \Gamma(\mathbb{R}^3)$. Here, the linear projector $\mathcal{P} = \mathbf{g} - \mathbf{N} \otimes \mathbf{N} : \Gamma(\mathbb{R}^3) \rightarrow \Gamma(TM)$ projects $\mathbb{R}^3$ vectors onto the surface. In the discrete setting (cf. figure 1.1), the strain rate tensor $\tilde{\mathbf{E}}_\alpha^{kj}$ and Killing operator $\mathcal{K}_{\alpha l}^{kji}$ are expressed in terms of the velocity U_j^i , projector $\mathcal{P}_\alpha^{ij} = \delta_\alpha^{ij} - \mathbf{N}_\alpha^i \mathbf{N}_\alpha^j$, and $\mathbb{R}^3$ component-wise surface

derivative $(\nabla_{\mathcal{P}})^i_{\alpha j}$ as:

$$2\tilde{E}_{\alpha}^{kj} = \sum_{l,i} 2\mathcal{K}_{\alpha l}^{kji} U_l^i = \sum_{l,i} \left(\mathcal{P}_{\alpha}^{ik} (\nabla_{\mathcal{P}})^j_{\alpha l} + \mathcal{P}_{\alpha}^{ij} (\nabla_{\mathcal{P}})^k_{\alpha l} \right) U_l^i. \quad (1.11)$$

Component-wise, $(\nabla_{\mathcal{P}})^i_{\alpha j} = \nabla^i \Phi_{\alpha j}$ follows scalar finite element discretization, where $\Phi_{\alpha j}$ is the hat function at vertex j restricted to face α . The divergence (1.4) is given by the trace $\text{DIV}_{\alpha l}^i = \sum_j \mathcal{K}_{\alpha l}^{iji}$. The gradient (1.5) and covariant divergence (1.6) can be obtained, up to a sign flip, by permuting the lower indices of the divergence and Killing operators, respectively.

1.3.2 Variational integrator by Onsager's variational principle

Given the vertex position $\boldsymbol{\varphi}_{(0)}$ and pressure $p_{(0)}$ at the current time $t = 0$, with a time step $\epsilon > 0$, the velocity is discretized as $\mathbf{U} = \dot{\boldsymbol{\varphi}}/\epsilon = (\boldsymbol{\varphi} - \boldsymbol{\varphi}_{(0)})/\epsilon$, and the power input as $\dot{V}_{\boldsymbol{\varphi}} = (V_{\boldsymbol{\varphi}} - V_{\boldsymbol{\varphi}_{(0)}})/\epsilon$. State variables $\boldsymbol{\varphi}_{(\epsilon)}$ and $p_{(\epsilon)}$ at $t = \epsilon$ follow a time-incremental Onsager's variational principle on a discrete Rayleighian (cf. §1.2.3), $\boldsymbol{\varphi}_{(\epsilon)}, p_{(\epsilon)} = \arg \min_{\boldsymbol{\varphi}} \max_p \mathcal{R}$, where

$$\begin{aligned} \mathcal{R}(\boldsymbol{\varphi}, p; \boldsymbol{\varphi}_{(0)}, p_{(0)}, \epsilon) &:= \epsilon^2 \left[\mu \|\tilde{\mathbf{E}}\|^2 + \langle \text{DIV } \mathbf{U}, p \rangle + \dot{V}_{\boldsymbol{\varphi}} - \langle \mathbf{U} | \mathbf{B} \rangle \right] \\ &\equiv \mu \mathcal{K}_{\lambda m}^{kji} (A_{\lambda \gamma})^{-1} \mathcal{K}_{\gamma n}^{kjl} \dot{\varphi}_m^i \dot{\varphi}_n^l + \epsilon (\dot{\varphi}_i^l \text{DIV}_{\alpha i}^l p_{\alpha} + V_{\boldsymbol{\varphi}} - \dot{\varphi}_i^l B_i^l) \quad (1.12) \\ &\equiv \mu \dot{\boldsymbol{\varphi}}^{\top} \mathbf{L} \dot{\boldsymbol{\varphi}} + \epsilon \left[(\text{DIV } \dot{\boldsymbol{\varphi}})^{\top} p + V_{\boldsymbol{\varphi}} - \dot{\boldsymbol{\varphi}}^{\top} \mathbf{B} \right]. \end{aligned}$$

Here, we assume Einstein summation and $\mathbf{L} = \mathcal{K}^{\top} \mathbf{A}^{-1} \mathcal{K}$ is half of the discrete viscosity Laplacian (cf. §1.2.3), where $A_{\lambda \gamma}$ is the area of the face γ for $\lambda = \gamma$ and zero otherwise.

Under any choice of $V_{\boldsymbol{\varphi}}$ (assuming $\mathbf{B} = \mathbf{0}$), the variational integrator is unconditionally stable by design, as it preserves the system's dissipative structure. The dynamics follow a gradient flow of $V_{\boldsymbol{\varphi}}$ in the metric space defined by the Stokes operator, ensuring a monotonic decrease in the Rayleighian: $\mathcal{R}_{(\epsilon)} \leq \mathcal{R}_{(0)}$. Therefore,

the allowable time step size depends only on the solvability of the optimization in (1.12), which can be efficiently handled using standard numerical optimization methods. Here we use a simple gradient flow to solve this saddle-point optimization: $H\dot{\boldsymbol{\varphi}} = -\delta_{\boldsymbol{\varphi}}\mathcal{R} = -2\mu\mathbf{L}\dot{\boldsymbol{\varphi}} - \epsilon(\text{DIV}^\top p + \delta_{\boldsymbol{\varphi}}V_{\boldsymbol{\varphi}} - \mathbf{B})$ and $h\dot{p} = \delta_p\mathcal{R} = \text{DIV}\dot{\boldsymbol{\varphi}}$, where H and h provide a general metric for the gradient flow⁴. In §1.4, we adopt the Helfrich Hamiltonian as V and follow the discretization used in Zhu et al. [130].

As mentioned in §1.2.3, equation (1.10) and its discrete version (1.12) have non-unique solution up to rigid body modes in $\mathbb{R}^3$. To ensure that we obtain continuous dynamics during time evolution, we consistently project out rigid body modes.

1.4 Results

In this section, we validate the discrete model, analyze its convergence properties, and demonstrate its applicability to manifolds with intricate geometries and topologies. Numerical validation against analytical solutions is presented in §1.4.1, with further examples provided in §1.4.2 and §1.4.3 to showcase its effectiveness.

1.4.1 Validations

We validate the differential operators $\mathcal{K}$ and GRAD on a spheroid with semi-axes a and c , parametrized by latitude $\beta \in [-\pi/2, \pi/2]$ and longitude $\theta \in (-\pi, \pi]$, using the realization $\boldsymbol{\varphi}(\beta, \theta) = [a \cos \beta \cos \theta, a \cos \beta \sin \theta, c \sin \beta]$.

In figure 1.2(a), we test the operator $\mathcal{K}$ by computing the lowest ten eigenvalues of $\mathbf{L} = \mathcal{K}^\top \mathbf{A}^{-1} \mathcal{K}$ with $\mathbf{L}\mathbf{U}_i = \lambda_i \mathbf{A}\mathbf{U}_i$ (cf. (1.12)). As predicted by the continuous theory, there are six vanishing eigenvalues, forming a six dimensional kernel of $\mathcal{K}$ corresponding to the space of rigid body motions in $\mathbb{R}^3$.

To validate the operator GRAD (1.16), we check that the numerical evaluation of

⁴We provide a MATLAB implementation available at <https://t.ly/vUxfh>. The tensorial calculations in (1.11) and (1.12) rely on `sptensor` [9]. Remeshing is performed using SideFx Houdini.

$A^{-1}|\text{DIV}^\top 1|$ aligns with the analytical mean curvature $H(\beta)$ of an oblate spheroid with $a = 1$ and $c = 0.5$, as shown in figure 1.2(b). In fact, this evaluation of mean curvature is identical with the established cotan Laplacian discretization [68].

We evaluate the convergence of our discretization of (1.10) by comparing it to an analytical solution on a unit sphere. The forcing term is analytically prescribed as $\mathbf{B} = \mathbf{b} + b_n \mathbf{N} = \mathbf{N} \times \text{grad } \phi + b_n \mathbf{N}$, where ϕ is a spherical harmonic satisfying $\Delta \phi := \text{div grad } \phi = \lambda \phi$, $\mathbf{N}$ is the surface normal, and b_n is a constant. On the unit sphere, $\mathbf{b}$ is an eigenfunction of the viscosity Laplacian with eigenvalue $\lambda + 2$. The analytical solution $\bar{\mathbf{U}} = -\mathbf{b}/(\lambda + 2)$ and $\bar{p} = -b_n/2$ satisfies (1.10) ($\mu = 1$, $\delta_\phi V_\phi = 0$) exactly (cf. (1.19)). Numerically, the relative residual of the momentum equation, $\varepsilon_U = \|2A^{-1}L\bar{\mathbf{U}} - A^{-1}\text{DIV}^\top \bar{p} - \mathbf{B}\|/\|\mathbf{B}\|$, and the residual of the continuity equation, $\varepsilon_p = \|A^{-1}\text{DIV} \bar{\mathbf{U}}\|$, shown in figure 1.2(c), decrease approximately linearly as the mesh is refined.

Lastly, figure 1.2(d) illustrates the total area of the fluid film during the Stokes relaxation ($\mu = 1$) from a prolate spheroid ($a = 1$, $c = 2$) to a sphere under Helfrich energy ($\kappa = 0.05$), without mesh reparametrization. Local incompressibility conserves the global area within a relative error of 10^{-5} throughout the evolution.

1.4.2 Relaxation of a genus-6 torus

We model the evolution of a high-genus, non-analytical lipid membrane ($\mu = 1$, $\kappa = 0.05$) based on the Helfrich-Stokes relaxation in figure 1.3(a). In theory, the elastic Helfrich Hamiltonian should be monotonically dissipated through viscosity, $V + E_\mu = \int \kappa H^2 dA + \int_0^t \langle\langle \mathbf{T} | \mathbf{E} \rangle\rangle / 2 |_\tau d\tau = \text{const}$. Although our numerical results in figure 1.3(b) do not exactly match this theoretical expectation, they show very good agreement.

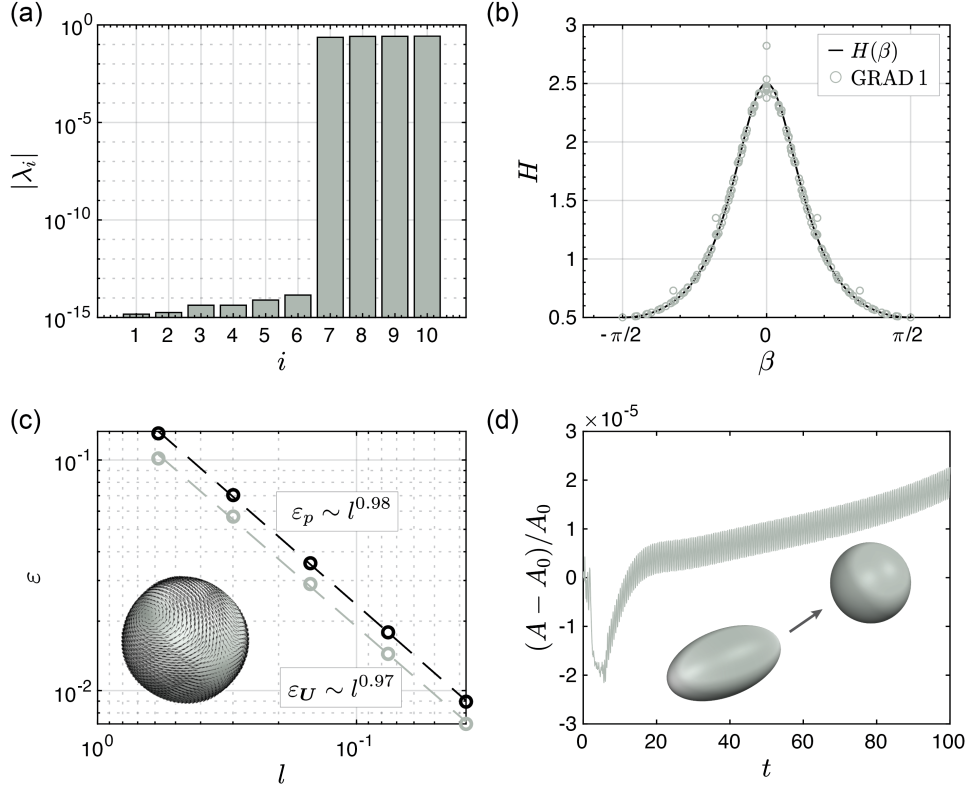


Figure 1.2. Validation of our model and method: (a) Minimum eigenvalues $\lambda_{1 \leq i \leq 10}$ of $\mathbf{L}\mathbf{U}_i = \lambda_i \mathbf{A}\mathbf{U}_i$; (b) Comparison of numerical GRAD 1 and analytical $H(\beta)$ evaluations of the mean curvature on a spheroid; (c) Relative error ε_U for the momentum equation and ε_p for the continuity equation as functions of mean edge length l , with the inset showing the tangential forcing $\mathbf{b}$ constructed from a spherical harmonic; (d) Relative error in total area A compared to the initial area A_0 during Helfrich-Stokes relaxation, with insets showing the initial frame (prolate spheroid) and the final frame (sphere) of the relaxation.

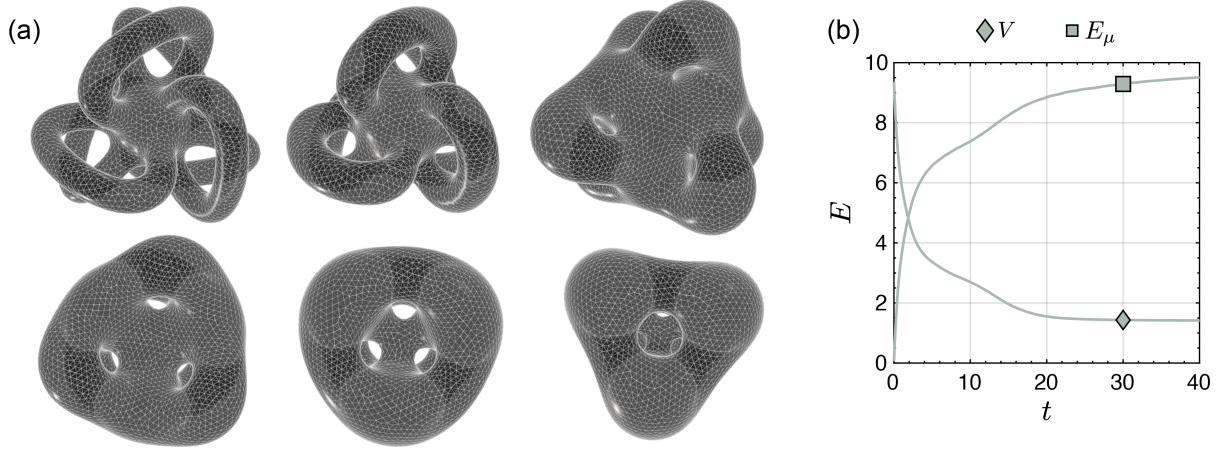


Figure 1.3. (a) Helfrich-Stokes relaxation of a genus-6 torus. Snapshots are taken at $t = 0, 2, 8, 14, 20$, and 40 , from left to right, top to bottom. The animated simulation is available at https://youtu.be/Llh0_N0hCPw and in Movie 1 of the Supplementary Material. (b) Elastic energy $V = \int \kappa H^2 dA$ and cumulative dissipation $E_\mu = \int_0^t \langle \langle \mathbf{T} | \mathbf{E} \rangle \rangle / 2 |_\tau d\tau$, which theoretically sum to a constant value.

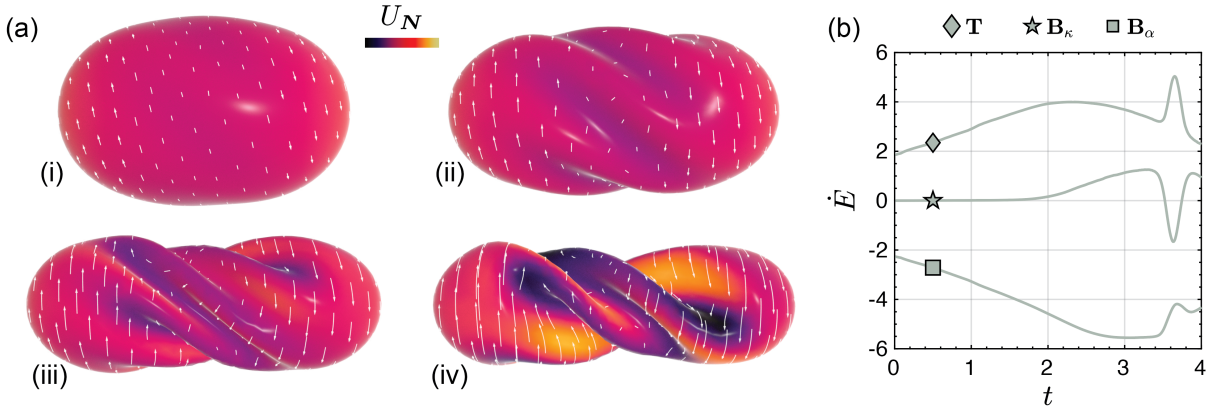


Figure 1.4. (a) Deformation of a lipid membrane under bulk flow, with the tangent velocity $\mathbf{u}$ shown as streamlines and the normal velocity U_N displayed in a colormap ranging from -1 to 1 (cf. Appendix A for the tangent-normal decomposition). Snapshots (i)-(iv) are taken at $t = 1.5, 2.5, 3.4$, and 3.8 , respectively. The animated simulation is available at <https://youtu.be/M0WsLihzRJk> and in Movie 2 of the Supplementary Material. (b) Energy dissipation rate $\dot{E}$ of the system due to viscosity $\dot{E}_\mu = \langle \langle \mathbf{T} | \mathbf{E} \rangle \rangle / 2$, elasticity $\dot{E}_\kappa = \langle \langle \mathbf{B}_\kappa | \mathbf{U} \rangle \rangle$, and bulk friction $\dot{E}_\alpha = \langle \langle \mathbf{B}_\alpha | \mathbf{U} \rangle \rangle / 2$.

1.4.3 Deformation of a lipid membrane under bulk flow

We model the deformation of a spherical lipid vesicle ($\mu = 1, \kappa = 0.01$) under the friction ($\alpha = 0.5$, cf. §1.2.3) from a constant bulk shear-extension flow defined by $\mathbf{U}_0(\theta, r, z) = \mathbf{U}_0^{\text{shear}} + \mathbf{U}_0^{\text{ext}} = (5rz)\hat{\mathbf{e}}_\theta + (-0.5r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z)$ in cylindrical coordinates. As illustrated in figure 1.4(a), this bulk flow stretches and twists the initially spherical vesicle along the z -direction. A snapthrough-like instability occurs around $t = 3.7$ between frame (iii) and frame (iv) in figure 1.4(a). While the bulk friction governs most of the dynamics, the Helfrich energy dominates during this buckling event, as indicated by the dip of the energy dissipation rate in figure 1.4(b) around $t = 3.7$.

1.5 Conclusion

Using Onsager's variational principle, we derived and reformulated the evolving Stokes equations on a manifold. We replaced the classic system of equations expressed in tangent-normal splitting with a simple, coordinate-free differential-geometric formulation. This approach directly leads to a straightforward discrete model and a numerical scheme to solve this long-standing problem in its full geometric generality.

Although the theoretical framework is not limited, the current minimal implementation is restricted to closed manifolds and employs first-order discretization. Future work could incorporate boundary conditions and higher-order discretizations. For applications, we expect the minimal system presented here to serve as a foundation for integrating more complex models for specific biophysical problems. These could include additional global volumetric/areal constraints, heterogeneity in material properties, in-plane anisotropy, and surface activity, as exemplified in Zhu et al. [130] and Chap. 3.

Appendix

A Evolving Stokes equations through normal-tangent decomposition

A.1 Differential operators for an evolving surface

The second fundamental form (i.e., curvature tensor), $\mathbb{I} \in \Gamma(T^*M \otimes_{\text{symm}} T^*M)$, is given by the derivative of the surface normal N :

$$\mathbb{I}[\mathbf{v}, \mathbf{w}] = (F\mathbf{v}) \cdot (\nabla_{F\mathbf{w}} N) =: \langle \mathbf{v}, S\mathbf{w} \rangle, \quad (1.13)$$

where $S : \Gamma(TM) \rightarrow \Gamma(TM)$ is the shape operator.

With the induced metric I , there is an intrinsic Levi-Civita covariant derivative $\nabla : \Gamma(TM) \rightarrow \Gamma(TM \otimes T^*M)$ on M . The covariant derivative in W used in (1.13) and (1.1) can be expressed in terms of the intrinsic covariant derivative and curvature tensor as:

$$\nabla_{F\mathbf{w}} F\mathbf{v} = \nabla_{\mathbf{w}} \mathbf{v} - \mathbb{I}[\mathbf{w}, \mathbf{v}]N. \quad (1.14)$$

We can decompose the velocity into tangent and normal parts, $\mathbf{U} = F\mathbf{u} + U_N N$, where $\mathbf{u} \in \Gamma(TM)$, $U_N \in C^\infty(W)$. Since $F\mathbf{u}$ effectively embeds the tangent vector $\mathbf{u}$ in $\mathbb{R}^3$, we will abbreviate $F\mathbf{u}$ as $\mathbf{u}$ when unambiguous. Given $\mathbf{v} \in T_p M$, (1.1) can be expressed as $\nabla_{F\mathbf{v}} \mathbf{U} = \nabla_{\mathbf{v}} \mathbf{u} + U_N S\mathbf{v} + N (\langle \text{grad } U_N, \mathbf{v} \rangle - \mathbb{I}[\mathbf{u}, \mathbf{v}])$. Combining with (1.2) and (1.3), we have $2(\mathcal{K}\mathbf{U})[\mathbf{v}, \mathbf{w}] = \langle \nabla_{\mathbf{v}} \mathbf{u}, \mathbf{w} \rangle + \langle \nabla_{\mathbf{w}} \mathbf{u}, \mathbf{v} \rangle + 2U_N \mathbb{I}[\mathbf{v}, \mathbf{w}]$, or

$$2\mathcal{K}\mathbf{U} = \nabla \mathbf{u} + (\nabla \mathbf{u})^\top + 2U_N \mathbb{I} =: 2\mathcal{K}\mathbf{u} + 2U_N \mathbb{I}, \quad (1.15)$$

where $\mathcal{K} : \Gamma(TM) \rightarrow \Gamma(T^*M \otimes_{\text{symm}} T^*M)$ denotes the intrinsic Killing operator. Com-

binning (1.4), (1.5) and (1.15), the divergence and gradient can be expressed as

$$\text{DIV } \mathbf{U} = \text{div } \mathbf{u} + 2HU_N, \quad \text{GRAD}(p) = \text{grad } p - 2pHN, \quad (1.16)$$

where $\text{div} = \text{tr}(\circ \mathcal{K})$ is the intrinsic divergence, $\text{grad} = -\text{div}^*$ is the intrinsic gradient, and $H = \text{tr } \mathbb{I} / 2$ is the mean curvature of the surface. From (1.6) and (1.15), we get:

$$\text{DIV}^\nabla \mathbf{T} = \text{div}^\nabla \mathbf{T} - \langle \mathbb{I} | \mathbf{T} \rangle \mathbf{N}, \quad (1.17)$$

where $\text{div}^\nabla = -\mathcal{K}^* : \Gamma(TM \otimes TM) \rightarrow \Gamma(T^*M)$ is the intrinsic covariant divergence.

A.2 Evolving Stokes equations

The viscosity Laplacian for an evolving surface can be explicitly decomposed as:

$$\begin{aligned} 2 \text{DIV}^\nabla \mathcal{K} \mathbf{U} &= 2 \left[\text{div}^\nabla \mathcal{K} \mathbf{u} + \text{div}^\nabla (U_N \mathbb{I}) - \left(\langle \mathbb{I}, \mathcal{K} \mathbf{u} \rangle + U_N |\mathbb{I}|^2 \right) \mathbf{N} \right] \\ &= (\Delta + K + \text{grad} \circ \text{div}) \mathbf{u} + 2S(\text{grad } U_N) + 4U_N \text{grad } H \\ &\quad - 2 \left[\langle \nabla \mathbf{u}, \mathbb{I} \rangle + U_N (4H^2 - 2K) \right] \mathbf{N}, \end{aligned} \quad (1.18)$$

where $\Delta = \text{div}^\nabla \nabla : \Gamma(TM) \rightarrow \Gamma(T^*M)$ is the intrinsic Bochner Laplacian, and $K = \det \mathbb{I}$ is the Gaussian curvature.

To summarize, by substituting (1.15)-(1.18) into (1.10), we get the system of evolving Stokes equations subject to external force $\mathbf{B} = \mathbf{b} + b_n \mathbf{N}$ and bending force $\mathbf{B}_\kappa = \kappa \mathbf{N} (\Delta H + 2H(H^2 - K))$:

$$\begin{aligned} \mu [(\Delta + K) \mathbf{u} + 2Q(\text{grad } U_N) + 2U_N \text{grad } H] - \text{grad } p + \mathbf{b} &= 0, \\ -2\mu \left[\langle \nabla \mathbf{u}, \mathbb{I} \rangle + U_N (4H^2 - 2K) \right] + 2pH + b_n + \kappa (\Delta H + 2H(H^2 - K)) &= 0, \\ \text{div } \mathbf{u} + 2HU_N &= 0. \end{aligned} \quad (1.19)$$

Note that the Hopf differential $Q = S - HI$ is the traceless part of the shape operator S that represents the deviatoric curvature. Formulation (1.19) agrees with earlier results by Scriven [108] and Arroyo & DeSimone [6].

Chapter 2

Viscous vortex dynamics on curved surfaces

This chapter derives a complete vorticity-based formulation of the Navier–Stokes equations on general surfaces. The focus here is on finite-inertia fluid mechanics on stationary surfaces, while generalizing the formulation to surfaces with nontrivial topology, non-orientable structure, and boundaries possessing intrinsic curvature. This chapter is based on the paper *Viscous Vortex Dynamics on Surfaces* by Cuncheng Zhu, Hang Yin, and Albert Chern. The dissertation author and Hang Yin were co-primary investigators and co-lead authors of this work.

2.1 Introduction

In this paper, we derive the vorticity equation for the Navier–Stokes equation on general surfaces M . Even for simply-connected domains, few existing methods in the graphics literature have correctly incorporated the viscous terms. A commonly overlooked detail is that the *Gaussian curvature* K of the surface contributes an additional term to the viscous force in the Navier–Stokes equation. In the vorticity formulation,

this leads to the expression:

$$\underbrace{\frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w}_{\text{previous model}} = \nu \Delta w + \underbrace{2\nu \operatorname{curl}(K\mathbf{u})}_{\text{curvature term}} \quad (2.1)$$

where w is the scalar vorticity, $\mathbf{u}$ the velocity field, Δ the Laplace–Beltrami operator, and ν the kinematic viscosity. Furthermore, on non-simply-connected domains, curvature and viscosity also affect the evolution of the *harmonic components* of the velocity field. If $\mathbf{h}$ is a harmonic vector field and $\mathbf{u}$ is the velocity field, the coefficient $c = \langle\langle \mathbf{h}, \mathbf{u} \rangle\rangle$ evolves according to

$$\frac{dc}{dt} = \underbrace{\int_M [(\mathbf{h} \times \mathbf{u})w]}_{\substack{\text{inviscid part} \\ [127]}} + \underbrace{2\nu(\mathbf{h} \cdot \mathbf{u})K}_{\text{new viscous terms}} dA + \oint_{\partial M} \nu h_{\partial} w ds, \quad (2.2)$$

where h_{∂} denotes the tangential component of $\mathbf{h}$ along the boundary. We show that these geometric terms are essential for capturing physically accurate behavior of viscous flows on curved surfaces, such as the conservation of frictionless rigid body motions.

Our proposed formulation and the accompanying numerical solver apply to general surfaces, including *non-orientable* ones (Figure 2.12). The framework supports a variety of boundary conditions, including the commonly used *no-slip*, *Navier friction*, and *free-slip* boundary conditions.

We highlight several additional findings:

- We analyze the effect of curvature term in (2.1) and show that a *curvature sheet* can maintain a *jump discontinuity* in vorticity, even in the presence of the diffusion term $\nu \Delta w$. We derive an explicit formula for this jump condition (Theorem 2.2.3).
- We discover a geometric equivalence: setting the friction coefficient is effectively equivalent to modifying the *geodesic curvature* of the boundary. Based on this

principle, imposing boundary friction corresponds to adjusting the boundary's intrinsic angles (Sections 2.3.2 and 2.5.3).

- We introduce the concept of combining Gaussian curvature and the boundary geodesic curvature into a unified *Gauss–Bonnet curvature density*. This reveals that the boundary curvature affects vorticity and harmonic components in the same way as Gaussian curvature. Notably, the curvature terms in (2.1) and (2.2) remain important even on flat domains with curved boundaries (Section 2.3.1). This insight also enables an elegant and stable discretization for incorporating vortex phenomena at the boundary.
- This curvature effect *directly gives rise to the Kutta condition*—a classical result in aerodynamics stating that flow must detach cleanly at the sharp trailing edge of an airfoil. Traditionally, this behavior is attributed to frictional boundary layers. However, we demonstrate that our formulation reproduces the Kutta condition naturally under *frictionless free-slip boundaries* without explicitly modeling boundary layers (Remark 2.3.1, Figure 2.4).

We present a detailed derivation of the Navier–Stokes equation on surfaces and its vorticity formulation in Section 2.2. The derivation of the boundary conditions and the extension to non-orientable surfaces are given in Section 2.3 and Section 2.4 respectively. Our numerical algorithm is described in Section 2.5, followed by simulation results in Section 2.6.

2.2 Equations of motion

The goal of this section is to derive the vorticity formulation of the Navier–Stokes equation on surfaces. In Sections 2.2.1 and 2.2.2, we clarified the appropriate form of the Navier–Stokes equation for surfaces. We then derive the corresponding vorticity equation, along with the accompanying harmonic field equation, in Section 2.2.3.

Let the fluid domain M be a two-dimensional Riemannian manifold. Here we assume that M is oriented while leaving the discussion of non-orientable domains to Section 2.4.

Let $\mathfrak{X} := \Gamma(TM)$ denote the space of tangent vector fields. Define the subspace $\mathfrak{X}_{\text{div}} \subset \mathfrak{X}$ as the collection of divergence-free vector fields with no-penetration boundary condition:

$$\mathfrak{X}_{\text{div}} := \{v \in \mathfrak{X} \mid \text{div } v = 0 \text{ in } M, \langle v, n \rangle = 0 \text{ on } \partial M\}. \quad (2.3)$$

The space $\mathfrak{X}_{\text{div}}$ is the feasible space for the velocity field of an incompressible flow on M . Here, n is the unit vector at ∂M that is tangent to M and inward normal to ∂M .

At each time t , the fluid velocity is a vector field $u_t \in \mathfrak{X}_{\text{div}}$. We will suppress the subscript t for these time dependent variables.

2.2.1 Incompressible Navier–Stokes equation

The *incompressible Navier–Stokes equation* is given by

$$\frac{\partial}{\partial t} u + \nabla_u u = -\text{grad } p + \text{div } \tau \quad (2.4)$$

where p is a scalar function representing the pressure or the Lagrange multiplier that keeps u in the subspace $\mathfrak{X}_{\text{div}}$, and $\tau \in \Gamma(\odot^2 TM)$ is a symmetric bivector (two-vector) tensor representing viscous stress.¹ Here, ∇ is the covariant derivative (a.k.a. the Levi-Civita connection) for the tangent bundle.

The divergence ($\text{div } \tau$) of a symmetric bivector τ also depends on the Levi-Civita connection. It represents the net force arising from the stress τ . In coordinate index notation, $(\text{div } \tau)^i = \nabla_j \tau^{ij}$.

¹The notation $\odot$ for the tensor bundle $\odot^2 TM = TM \odot TM$ is the symmetric tensor product. In particular, $\odot^2 TM \subset \otimes^2 TM$ is the bundle for symmetric tensors of bivector type.

To close the system (2.4), one must model how the viscous stress tensor τ depends on the variable $\mathbf{u}$. For Newtonian fluids, the viscous stress tensor τ is a linear function of the velocity gradient $\nabla \mathbf{u}$ in a particular form. Within the tensor $\nabla \mathbf{u}$ there is a component that quantifies the rate of shearing or non-isometric deformations the flow experiences. This quantity is called the *strain rate*. The viscous stress tensor is modeled to be proportional to this quantity. This model will be stated in subsubsection 2.2.1.

Strain rate

The *symmetric part* of the velocity gradient corresponds to the rate of non-isometric deformation. This symmetrized gradient has a natural geometric definition, given as follows:

Definition 2.2.1 (Killing operator). *Let $g \in \Gamma(\odot^2 T^*M)$ denote the metric tensor on M . The **Killing operator** is the differential operator $\mathcal{K}: \Gamma(TM) \rightarrow \Gamma(\odot^2 T^*M)$ acting on vector fields, defined as half the Lie derivative of the metric tensor along the vector field:*

$$\mathcal{K}\mathbf{u} := \frac{1}{2} \mathcal{L}_{\mathbf{u}} g. \quad (2.5)$$

We now expand the definition of the Lie derivative. For a velocity field $\mathbf{u}$, the tensor $\mathcal{L}_{\mathbf{u}} g$ represents the rate of change of the pullback metric under the flow generated by $\mathbf{u}$. Explicitly, $\mathcal{L}_{\mathbf{u}} g = \frac{\partial}{\partial t}|_{t=0}(\varphi_t^* g)$, where φ_t is the flow map generated by $\mathbf{u}$. In other words, $\mathcal{K}\mathbf{u}$ directly measures the rate of non-isometric deformation induced by the flow of $\mathbf{u}$.

The following proposition shows that the strain rate $\mathcal{K}\mathbf{u}$ indeed recovers the symmetrized velocity gradient.

Proposition 2.2.1. *The symmetric bilinear form $\mathcal{K}\mathbf{u} \in \Gamma(\odot^2 T^*M)$ for a vector field $\mathbf{u} \in \Gamma(TM)$*

$\Gamma(TM)$ can be explicitly expressed as follows. At each point $p \in M$, for all $\mathbf{a}, \mathbf{b} \in T_p M$:

$$(\mathcal{K}\mathbf{u})[[\mathbf{a}, \mathbf{b}]] = \frac{1}{2} (\langle \nabla_{\mathbf{a}} \mathbf{u}, \mathbf{b} \rangle + \langle \nabla_{\mathbf{b}} \mathbf{u}, \mathbf{a} \rangle). \quad (2.6)$$

Proof. Section A.1. □

Remark 2.2.1. In an orthonormal basis, equation (2.6) shows that $\mathcal{K}\mathbf{u}$ corresponds to the 2-by-2 matrix $\mathcal{K}\mathbf{u} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$. That is, $\mathcal{K}\mathbf{u}$ is the symmetric part of the matrix $\nabla \mathbf{u}$, familiar to readers with a continuum mechanics background. However, the expression $(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ informally combines tensors of different types: $\nabla \mathbf{u}$ sends vectors to vectors, while $\nabla \mathbf{u}^\top$ sends covectors to covectors. Writing $(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ precisely requires an additional layer of metric conversions. For clarity, we will therefore continue working with the invariant object $\mathcal{K}\mathbf{u}$.

Definition 2.2.2 (Killing vector field). A vector field $\mathbf{v}$ is called a **Killing vector field** if $\mathcal{K}\mathbf{v} = 0$.

Killing vector fields generate isometric flows—rigid motions that do not involve shearing. Domains with continuous symmetries, such as surfaces of revolution, admit Killing vector fields. If the fluid velocity is a Killing field, then the fluid experiences no internal viscous stress.

Constitutive model

The viscous stress in (2.4) is proportional to the strain rate

$$\boldsymbol{\tau} = 2\nu(\mathcal{K}\mathbf{u})^\sharp. \quad (2.7)$$

where $\nu \geq 0$ is the kinematic viscosity coefficient. Here, the $\sharp$ operator in $(\mathcal{K}\mathbf{u})^\sharp$ denotes the isomorphism $\odot^2 T^*M \rightarrow \odot^2 TM$ with respect to the Frobenius norm for order-2 tensors. In index notation, $\tau^{ij} = 2\nu g^{ik} g^{j\ell} (\mathcal{K}\mathbf{u})_{k\ell}$.

The stress–strain relationship (2.7) can also be derived from variational principles

via the *Rayleigh dissipation functional*:

$$\mathcal{R}: \mathfrak{X}_{\text{div}} \rightarrow \mathbb{R}, \quad \mathcal{R}[\mathbf{u}] := \iint_M \nu |\mathcal{K}\mathbf{u}|^2 dA, \quad (2.8)$$

where $|\cdot|^2$ is the Frobenius norm. The dissipation functional (2.8) quantifies the rate of energy dissipation, modeled under the physical assumption that the dissipation depends quadratically and isotropically on the strain rate. The viscous force in the momentum equation (2.4) is then obtained via Rayleigh's variational principle: the viscous force is the negative gradient of the Rayleigh dissipation functional, added directly to the inviscid equation of motion (the Euler equation).

We recall the notion of a functional gradient:

Definition 2.2.3. *The gradient of a functional $\mathcal{F}: \mathfrak{X}_{\text{div}} \rightarrow \mathbb{R}$ at $\mathbf{u} \in \mathfrak{X}_{\text{div}}$, with respect to the L^2 inner product structure $\langle\langle \cdot, \cdot \rangle\rangle := \iint_M \langle \cdot, \cdot \rangle dA$ on $\mathfrak{X}_{\text{div}}$, is the vector field $(\text{grad } \mathcal{F})|_{\mathbf{u}} \in \mathfrak{X}_{\text{div}}$ satisfying*

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{F}[\mathbf{u} + \epsilon \dot{\mathbf{u}}] = \langle\langle \text{grad } \mathcal{F}|_{\mathbf{u}}, \dot{\mathbf{u}} \rangle\rangle \quad (2.9)$$

for any variation $\dot{\mathbf{u}} \in \mathfrak{X}_{\text{div}}$ that vanishes on ∂M .

Proposition 2.2.2. *The functional gradient $(\text{grad } \mathcal{R})$ of $\mathcal{R}$ defined in (2.8) is given by*

$$(\text{grad } \mathcal{R})|_{\mathbf{u}} = -2\nu \text{div}(\mathcal{K}\mathbf{u})^\sharp. \quad (2.10)$$

Proof. Section A.4. □

Summary

The Navier Stokes equation (2.4) with the stress–strain relation (2.7) becomes:

$$\boxed{\frac{\partial}{\partial t} \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} = -\text{grad } p + 2\nu \text{div}(\mathcal{K}\mathbf{u})^\sharp.} \quad (2.11)$$

This equation can be viewed as the Euler equation $\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\text{grad } p$, augmented by the additional gradient descent term $-(\text{grad } \mathcal{R})|_{\mathbf{u}}$ of the dissipation functional $\mathcal{R}$.

2.2.2 Hodge, Bochner, and viscous Laplacians

On a flat Euclidean domain, the divergence of the symmetrized gradient equals the vector Laplacian when applied to divergence-free fields, $\nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) = \Delta \mathbf{u} + \nabla(\nabla \cdot \mathbf{u})$, which leads to the familiar form of the Navier–Stokes equation $\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\text{grad } p + \nu \Delta \mathbf{u}$. However, this vector identity *does not* hold when the domain has curvature.

There are *three* relevant surface Laplacians for vector fields in this context, each defined via the functional gradient of a Dirichlet-type energy.

Dirichlet energies

Definition 2.2.4 (Standard Dirichlet energy). *The **standard Dirichlet energy** for a vector field $\mathbf{v} \in \Gamma(TM)$ is:*

$$\mathcal{E}_D[\mathbf{v}] := \frac{1}{2} \iint_M |\nabla \mathbf{v}|^2 dA. \quad (2.12)$$

Definition 2.2.5 (Hodge-Dirichlet energy). *The **Hodge-Dirichlet energy** for a vector field $\mathbf{v} \in \Gamma(TM)$ is:*

$$\mathcal{E}_H[\mathbf{v}] := \frac{1}{2} \iint_M (|d\mathbf{v}^\flat|^2 + |\delta \mathbf{v}^\flat|^2) dA, \quad (2.13)$$

where d and δ denote the exterior derivative and the codifferential respectively.

Note that if $\mathbf{v}$ is restricted to $\mathfrak{X}_{\text{div}}$ (divergence-free fields, where $\mathbf{v}^\flat$ is coclosed), the Hodge-Dirichlet energy simplifies to

$$\mathcal{E}_H[\mathbf{v}] = \frac{1}{2} \iint_M |d\mathbf{v}^\flat|^2 dA, \quad \mathbf{v} \in \mathfrak{X}_{\text{div}}. \quad (2.14)$$

Here, dv^b is the vorticity 2-form, corresponding to the *skew-symmetric part* of $2\nabla v$.

Definition 2.2.6 (Viscous-Dirichlet energy). *The **viscous-Dirichlet energy**—also called the **Killing energy** [11]—for a vector field $v \in \Gamma(TM)$ is:*

$$\mathcal{E}_V[v] := \iint_M |\mathcal{K}v|^2 dA. \quad (2.15)$$

The standard Dirichlet energy measures the smoothness of the vector field, a notion commonly used in vector field design in geometry processing. The Hodge-Dirichlet energy, when restricted to divergence-free fields, measures the L^2 -strength of the vorticity. The viscous-Dirichlet energy measures the rate of energy dissipation ($\mathcal{R} = \nu \mathcal{E}_V$; see (2.8)) and also quantifies the deviation of a vector field from being a Killing field.

Remark 2.2.2. *When M is a flat Euclidean domain (ignoring boundary effects), the three Dirichlet energies are equal. In fluid mechanics, they are all referred to as the **enstrophy** of the flow, with physical interpretations relating to both the vorticity magnitude and the level of dissipation. However, to our knowledge, there is no universal agreed-upon definition of enstrophy for fluid on a Riemannian manifold, where these two interpretations diverge.*

Vector Laplacians

The following are the three vector Laplacians associated with the three Dirichlet energies.

Definition 2.2.7. *The **Bochner Laplacian** (or **connection Laplacian**) for a vector field $v \in \Gamma(TM)$ is defined by*

$$\Delta_B v := -(\text{grad } \mathcal{E}_D)|_v. \quad (2.16)$$

The Bochner Laplacian satisfies $\Delta_B v = \text{div}(\nabla v)$ where div denotes the connection divergence (the negative L^2 -adjoint of ∇). An alternative expression is $\Delta_B v = \star^{-1} d^\nabla \star \nabla v$ where v is regarded as a vector-valued 0-form, and d^∇ is the covariant exterior derivative for vector-valued k -forms.

Definition 2.2.8. *The **Hodge Laplacian** for a vector field $v \in \Gamma(TM)$ is defined by*

$$\Delta_H v := -(\text{grad } \mathcal{E}_H)|_v. \quad (2.17)$$

The Hodge Laplacian takes the familiar form: $\Delta_H v = -((d\delta + \delta d)v)^\sharp$, standard in exterior calculus.

Definition 2.2.9. *The **viscous Laplacian** for a vector field $v \in \Gamma(TM)$ is defined by*

$$\Delta_V v := -(\text{grad } \mathcal{E}_V)|_v. \quad (2.18)$$

The viscous Laplacian is given by $\Delta_V v = 2 \text{div}(\mathcal{K}v)^\sharp$, where $\text{div} \circ \sharp$ is the negative adjoint of $\mathcal{K}$.

All three Laplacians are self-adjoint (with respect to L^2) and are negative semi-definite elliptic operators. Their differences become apparent when examining their kernels (zero eigenspaces). For simplicity, we restrict $v \in \mathfrak{X}_{\text{div}}$.

- $\Delta_B v = 0$ if and only if $v \in \mathfrak{X}_{\text{div}}$ is *parallel*. In particular, Δ_B has a nontrivial kernel only when the domain is flat.
- $\Delta_H v = 0$ if and only if $v \in \mathfrak{X}_{\text{div}}$ is *harmonic*. Thus, Δ_H has a kernel if and only if the domain is not simply connected.
- $\Delta_V v = 0$ if and only if $v \in \mathfrak{X}_{\text{div}}$ is a *Killing field*. Consequently, Δ_V has a kernel if and only if the metric domain admits continuous symmetries.

The kernel of Δ_B are parallel translational flows, which exist only in Euclidean whose

boundaries (if any) are straight and compatible with translation, such as in a pipe domain. In contrast, the existence of a kernel for Δ_H depends solely on the topology of the domain. Meanwhile, the existence of a kernel for Δ_V reflects the presence of domain symmetries. For example, Killing fields exist on simply connected curved domains such as the round sphere.

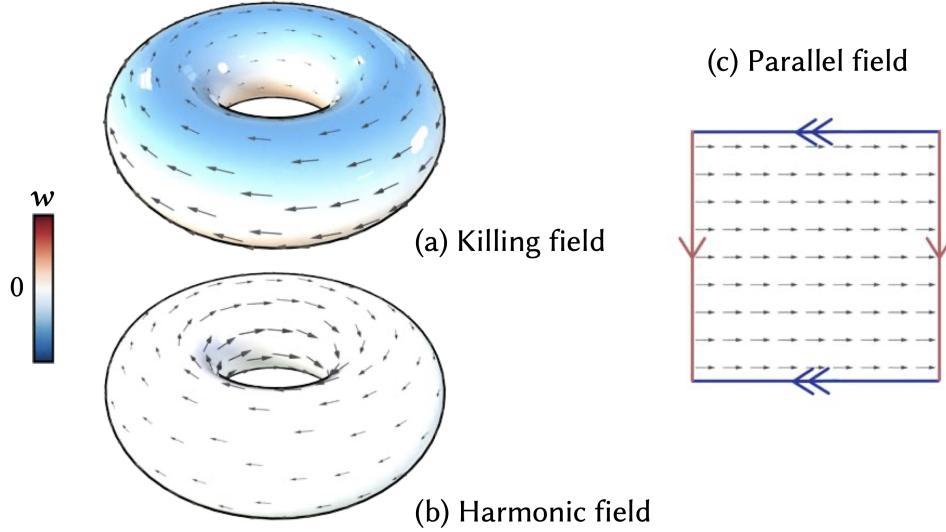


Figure 2.1. Kernels of the Laplacians Δ_V , Δ_H , and Δ_B on tori. (a) A vector field in $\ker(\Delta_V)$ is a Killing field, which generally has nonzero vorticity. (b) A vector field in $\ker(\Delta_H)$ is harmonic and therefore has zero vorticity. (c) A vector field in $\ker(\Delta_B)$ is parallel, which is possible on a flat torus; on this domain Δ_B , Δ_H , Δ_V share the same kernel.

The following theorem states the precise relationship between the three Laplacians.

Theorem 2.2.1 (Weitzenböck identity). *When applied to divergence-free vector fields $v \in \mathfrak{X}_{\text{div}}$ on a surface, the Bochner, Hodge, and viscous Laplacians are related by:*

$$\Delta_B v = \Delta_H v + K v = \Delta_V v - K v, \quad v \in \mathfrak{X}_{\text{div}}, \quad (2.19)$$

where K is the Gaussian curvature of the surface.

Proof. Section A.6. □

Summary

The Navier–Stokes equation can be written in terms of the different Laplacians as any of the following equivalent forms:

$$\boxed{\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\operatorname{grad} p + \nu\Delta_V\mathbf{u}} \quad (2.20)$$

$$\boxed{\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\operatorname{grad} p + \nu\Delta_B\mathbf{u} + \nu K\mathbf{u}} \quad (2.21)$$

$$\boxed{\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} = -\operatorname{grad} p + \nu\Delta_H\mathbf{u} + 2\nu K\mathbf{u}.} \quad (2.22)$$

A direct consequence of the Navier–Stokes equation is that if a Killing field exists, the component of the velocity field along the Killing field is conserved.

Theorem 2.2.2. *Suppose $\mathbf{v} \in \mathfrak{X}_{\operatorname{div}}$ is a Killing field, and let $\mathbf{u}$ be a solution to the Navier–Stokes equation. Then $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle$ remains constant in time.*

Proof. Section A.2. □

2.2.3 Vorticity formulation

We now derive the vorticity formulation of the Navier–Stokes equation.

Vorticity equation

Applying the $\flat$ operator to (2.22), we obtain the Navier–Stokes equation in covector form:

$$\frac{\partial}{\partial t}\mathbf{u}^\flat + \mathcal{L}_{\mathbf{u}}\mathbf{u}^\flat = -d\left(p - \frac{1}{2}|\mathbf{u}|^2\right) - \nu\delta d\mathbf{u}^\flat + 2\nu K\mathbf{u}^\flat. \quad (2.23)$$

Taking the exterior derivative of (2.23), and letting

$$\omega := d\mathbf{u}^\flat \in \Omega^2(M) \quad (2.24)$$

denote the vorticity 2-form, we obtain:

$$\frac{\partial}{\partial t}\omega + \mathcal{L}_{\mathbf{u}}\omega = -\nu d\delta\omega + 2\nu d(K\mathbf{u}^\flat). \quad (2.25)$$

Let the vorticity scalar function be defined by

$$w = \star^{-1}\omega \in \Omega^0(M). \quad (2.26)$$

Then (2.25) becomes

$$\frac{\partial}{\partial t}w + \mathcal{L}_{\mathbf{u}}w = \nu\Delta w + 2\nu\star^{-1}d(K\mathbf{u}^\flat), \quad (2.27)$$

where $\Delta = \star^{-1}d\star d$ is the standard (negative semi-definite) Laplace–Beltrami operator for scalar functions.

In vector calculus notation, this reads:

$$\boxed{\frac{\partial}{\partial t}w + \mathbf{u} \cdot \nabla w = \nu\Delta w + 2\nu \operatorname{curl}(K\mathbf{u}).} \quad (2.28)$$

Here, in two dimensions, we use the scalar curl operator $\operatorname{curl} := -\operatorname{div} \circ J$, where J denotes the 90° rotation operator.

Remark 2.2.3 (Effect of curvature on vorticity). *Equations (2.25) and (2.28) show that the vorticity evolves via a convection-diffusion process with an additional curvature-driven term $2\nu d(K\mathbf{u}^\flat)$. Applying the Leibniz rule $d(K\mathbf{u}^\flat) = dK \wedge \mathbf{u}^\flat + Kd\mathbf{u}^\flat$ we can expand the curvature term $2\nu \operatorname{curl}(K\mathbf{u})$ as:*

$$2\nu \operatorname{curl}(K\mathbf{u}) = 2\nu \langle -J\mathbf{u}, \operatorname{grad} K \rangle + 2\nu K\omega. \quad (2.29)$$

These terms can induce a vorticity confinement effect, opposing the diffusion $\nu\Delta w$, and can lead

to soliton-like behavior in the vorticity field.

To best illustrate this effect, consider the case where Gaussian curvature is concentrated along a curve segment $\Gamma \subset M$. Specifically, take $K = K_{\text{reg}} + f\delta_\Gamma$, where K_{reg} is a smooth function (the regular part of the curvature), and δ_Γ is the Dirac δ -distribution supported on Γ (a singular curvature sheet). The factor f is a smooth function defined along Γ . More precisely, for any test function $\varphi: M \rightarrow \mathbb{R}$, $\int_M K\varphi dA = \int_\Omega K_{\text{reg}}\varphi dA + \int_\Gamma f\varphi ds$. In this setting, the vorticity develops a jump discontinuity across Γ characterized by the following jump condition (see also Figure 2.2):

Theorem 2.2.3 (Curvature sheet and vorticity jump). *Under the vorticity equation (2.28) with $K = K_{\text{reg}} + f\delta_\Gamma$ and any $\nu > 0$, the vorticity w satisfies the jump condition*

$$[w]_\Gamma = 2f\langle \mathbf{u}, \mathbf{t}_\Gamma \rangle \quad (2.30)$$

where $\mathbf{t}_\Gamma$ is the tangent vector of Γ and $[w]_\Gamma$ denotes the jump of w across Γ in the direction $J\mathbf{t}_\Gamma$ normal to Γ . Specifically, for each $x \in \Gamma$, $[w]_\Gamma(x) = \lim_{\epsilon \searrow 0^+} w(\gamma(\epsilon)) - w(\gamma(-\epsilon))$, where $\gamma: (-a, a) \rightarrow M$ is any smooth curve with $\gamma(0) = x$ and $\gamma'(0) = J\mathbf{t}$.

Proof. Section B.1 □

Remark 2.2.4. The jump condition (2.30) is independent of the viscosity ν . This naturally leads to the question of whether, in the limit $\nu \rightarrow 0$, the jump vanishes abruptly or continuously? To analyze this limiting behavior, we regularize the curvature singularity by smoothing the curvature sheet with thickness ϵ . We now consider the limiting process involving both parameters ϵ and ν (see Figure 2.3 for a numerical example). For nonzero ϵ and ν , there exists a relaxation time τ for (2.30) to emerge after the initial condition. Dimension analysis shows that $\tau = O(\epsilon^2/\nu)$.² If $\epsilon^2 \rightarrow 0$ faster than $\nu \rightarrow 0$, the jump condition emerges, and the resulting jump is independent of the value of ν . In contrast, if $\epsilon > 0$ is fixed while $\nu \rightarrow 0$, then the jump vanishes

²By Buckingham's Π theorem, three variables τ, ϵ, ν , which involve only the dimensions of length and time, must be related through a dimensionless constant. In this case, the relation is $\nu\tau/\epsilon^2 = C$.

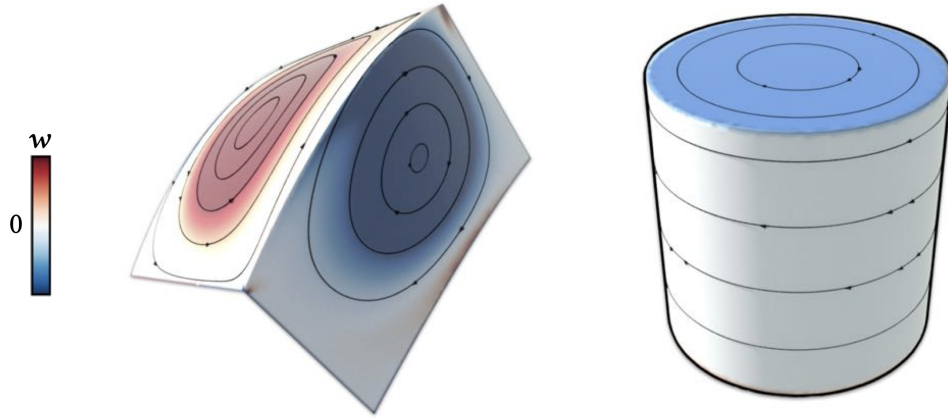


Figure 2.2. A singular sheet in the Gaussian curvature gives rise to a jump discontinuity in the vorticity, shown by the color (cf. Theorem 2.2.3). Left: Vorticity distribution in a simulation on a surface with a curved ridge. Right: The rigid body rotation field on a cylinder exhibits a vorticity jump that agrees with Theorem 2.2.3.

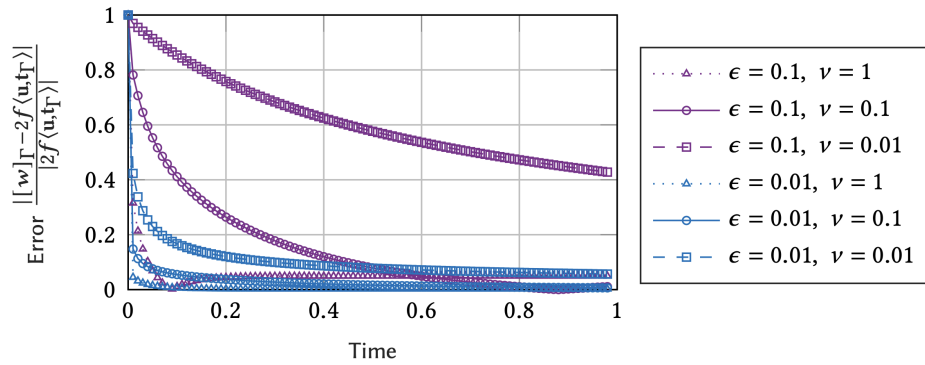


Figure 2.3. Validation of (2.30) with various viscosity ν on a cylinder of radius $R = 1/2$ (Figure 2.2, right) triangulated with averaged edge length ϵ . The Gaussian curvature in its regular and singular parts is given by $K = 0 + \frac{1}{R}\delta_{\Gamma}$ (i.e., $f = 2$ in (2.30)) where Γ is the rim of the cylinder, and the Dirac δ is smoothed with thickness $O(\epsilon)$. As $\epsilon \rightarrow 0$, the relative error of (2.30) rapidly reduces over time for all ν .

smoothly. The condition (2.30) is to be interpreted under the assumption $\epsilon = 0$, in which case the jump is independent of v . In practice, discretization introduces a finite $\epsilon \approx \text{edge length}$, and thus both parameters play a role in numerical results (Figure 2.3).

Velocity reconstruction

To complete (2.28), we describe how $\mathbf{u}$ is reconstructed from the variable w . This is achieved using a streamfunction and, when the domain is non-simply-connected, harmonic fields. This standard procedure is also detailed in [127, § 2.3–2.6 and § B] and [8, Eq. (1)].

Definition 2.2.10. Let $\mathfrak{H} := \{\mathbf{h} \in \mathfrak{X}_{\text{div}} \mid d\mathbf{h}^\flat = 0\} \subset \mathfrak{X}_{\text{div}}$ be the space of *harmonic vector fields*. Let $m := \dim(\mathfrak{H})$.

These harmonic vector fields are divergence-free and curl-free with the no-penetration boundary condition. The corresponding 1-forms $\mathcal{H}_C^1 := \mathfrak{H}^\flat$ are harmonic 1-forms satisfying the co-Dirichlet (Neumann) boundary condition: $d\mathbf{h}^\flat = 0, d \star \mathbf{h}^\flat = 0, j^* \star \mathbf{h}^\flat = 0$, where $j: \partial M \hookrightarrow M$ is the inclusion map of the boundary. The dimension m of $\mathfrak{H}$ is finite and equals the first Betti number of M , i.e., the dimension of its first homology group.

Let $\mathbf{h}_1, \dots, \mathbf{h}_m$ be a time-independent, L^2 -orthonormal basis for $\mathfrak{H}$. By the Hodge–Morrey–Friedrichs decomposition, any $\mathbf{u} \in \mathfrak{X}_{\text{div}}$ can be expressed as

$$\mathbf{u} = -J \text{grad } \psi + \sum_{i=1}^m c_i \mathbf{h}_i, \quad (\text{or } \mathbf{u}^\flat = \star^{-1} d\psi + \sum_{i=1}^m c_i \mathbf{h}_i^\flat), \quad (2.31)$$

where the *streamfunction* $\psi \in \Omega^0(M)$ is a scalar function satisfying a zero-Dirichlet boundary condition³ $\psi|_{\partial M} = 0$, and $c_i \in \mathbb{R}$ are time-dependent coefficients whose evolution will be discussed in subsection 2.2.3.

³In the literature of 2D streamfunction representation, the streamfunctions are often described as having constant Dirichlet boundary conditions, with different constants for each connected component of the boundary. These cases are handled by the harmonic part $\sum_{i=1}^m c_i \mathbf{h}_i$.

The streamfunction ψ is uniquely determined (up to an additive constant if $\partial M = \emptyset$) by the vorticity via the Poisson problem:

$$\begin{cases} -\Delta\psi = w & \text{in } M \\ \psi = 0 & \text{on } \partial M. \end{cases} \quad (2.32)$$

Evolution of the harmonic part

The coefficients c_i in (2.31) evolve over time [127]. The dynamics of c_i is nontrivial even in the inviscid case, derived in [127, §3]. In sum, the inviscid evolution of c_i is given by $\frac{dc_i}{dt} = \iint_M (\mathbf{h}_i \times \mathbf{u}) w dA$, which is obtained by taking the L^2 product of the Euler equation in *Lamb's form* $\frac{\partial}{\partial t} \mathbf{u}^\flat = -i_u \omega - d(p + \frac{1}{2}|\mathbf{u}|^2)$ [127, Eq. (23)] with the harmonic basis $\mathbf{h}_i$. Here, *Lamb's Euler equation* is obtained by expanding Cartan's formula $\mathcal{L}_u \mathbf{u}^\flat = i_u d\mathbf{u}^\flat + di_u \mathbf{u}^\flat = i_u \omega + d|\mathbf{u}|^2$, where i_u is the interior product.

Now, we derive the viscous case in a similar fashion.

First, we rewrite (2.23) in *Lamb's form*:

$$\frac{\partial}{\partial t} \mathbf{u}^\flat = -i_u \omega - d\left(p + \frac{1}{2}|\mathbf{u}|^2\right) - \nu \delta \omega + 2\nu K \mathbf{u}^\flat. \quad (2.33)$$

To derive the equation for $c_i = \langle \mathbf{u}, \mathbf{h}_i \rangle = \langle \mathbf{u}^\flat, \mathbf{h}_i^\flat \rangle$, we take the L^2 product between (2.33) and $\mathbf{h}_i^\flat$:

$$\frac{dc_i}{dt} = \iint_M \omega \llbracket \mathbf{h}_i, \mathbf{u} \rrbracket dA - \nu \langle \mathbf{h}_i^\flat, \delta \omega \rangle + 2\nu \iint_M \langle \mathbf{h}_i, \mathbf{u} \rangle K dA. \quad (2.34)$$

The middle term $\langle \mathbf{h}_i^\flat, \delta \omega \rangle$ can be expanded as

$$-\langle \mathbf{h}_i^\flat, \delta \omega \rangle = -\iint_M \mathbf{h}_i^\flat \wedge d\omega = \iint_M d(\omega \mathbf{h}_i^\flat) = \oint_{\partial M} \omega \mathbf{h}_i^\flat \quad (2.35)$$

using $\delta \omega = \star^{-1} d \star \omega$, $\langle \cdot, \cdot \rangle = \iint_M (\cdot) \wedge \star(\cdot)$, $d\mathbf{h}_i^\flat = 0$, $\star \omega = w$, and Stokes' theorem.

Thus, we obtain:

$$\boxed{\frac{dc_i}{dt} = \iint_M \left((\mathbf{h}_i \times \mathbf{u})w + 2\nu \langle \mathbf{h}_i, \mathbf{u} \rangle K \right) dA + \oint_{\partial M} \nu h_{i\partial} w ds} \quad (2.36)$$

where $h_{i\partial}$ is the tangential component of $\mathbf{h}_i$ along the boundary:

$$h_{i\partial}: \partial M \rightarrow \mathbb{R}, \quad h_{i\partial} := \langle \mathbf{h}_i, \mathbf{t} \rangle \quad (2.37)$$

with $\mathbf{t}$ being the unit tangent vector for the oriented curve ∂M . In two dimensions, the cross product $\times$ of two tangent vectors is a scalar, defined by $\mathbf{h}_i \times \mathbf{u} := \langle J\mathbf{h}_i, \mathbf{u} \rangle$.

Remark 2.2.5 (Interior and boundary viscous effects on the harmonic part). *If $\partial M = \emptyset$, the boundary term $\oint_{\partial M} \nu h_{i\partial} w ds$ vanishes. In this case, the only remaining viscous contribution to $\frac{dc_i}{dt}$ is the curvature term $\iint_M 2\nu \langle \mathbf{h}_i, \mathbf{u} \rangle K dA$. In general, the only interior viscous contribution to $\frac{dc_i}{dt}$ is the curvature term, while the diffusion term contributes solely through the boundary.*

Summary

The vorticity formulation for the Navier–Stokes equation can be summarized as follows. The state variables are the vorticity scalar field $w \in \Omega^0(M)$ and the coefficients $c_1, \dots, c_m \in \mathbb{R}$, all depending on time. Each state $(w, c_1, \dots, c_m)$ determines a unique velocity vector field $\mathbf{u}$ via (2.31), which requires solving the scalar Poisson problem (2.32) together with a fixed orthonormal basis $(\mathbf{h}_1, \dots, \mathbf{h}_m)$ for harmonic fields.

Theorem 2.2.4. *Under the Navier–Stokes equation (2.11), the variables $(w, c_1, \dots, c_m)$ evolve according to*

$$\boxed{\begin{cases} \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = \nu \Delta w + 2\nu \operatorname{curl}(K\mathbf{u}), \\ \frac{dc_i}{dt} = \iint_M [(\mathbf{h}_i \times \mathbf{u})w + 2\nu (\mathbf{h}_i \cdot \mathbf{u})K] dA + \oint_{\partial M} \nu h_{i\partial} w_{\partial} ds. \end{cases}} \quad (2.38)$$

We emphasize that the evolution system (2.38) remains incomplete until additional boundary conditions are specified.

Remark 2.2.6. *The evolution system (2.38) is not yet complete because certain variables on the right-hand side remain undetermined. Specifically, the boundary value $w|_{\partial M}$ of the vorticity is required. Note that the vorticity equation in (2.38) governs only the interior of M and provides no direct information about $w|_{\partial M}$. These boundary values are necessary both for evaluating the diffusion term $\nu \Delta w$ in the vorticity equation and for computing the boundary integral $\oint_{\partial M} \nu h_{i\partial} w_{\partial} ds$. The remaining components of the system, including (2.31) and (2.32), involve only the interior values of w .*

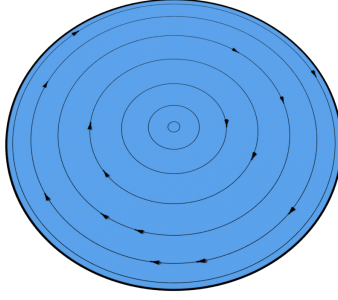
2.3 Boundary conditions

As discussed in Remark 2.2.6, the boundary vorticity $w_{\partial}: \partial M \rightarrow \mathbb{R}$ is an unknown function appearing on the right-hand side of the evolution equation (2.38). An additional set of equations is required to constrain the freedom of w_{∂} at each time t . In this section, we detail how three common boundary conditions—free-slip, friction, and no-slip—are incorporated into (2.38). In particular, we show that the curvature of the boundary plays a crucial role in these conditions.

2.3.1 Free-slip boundary

The free-slip boundary condition is the mathematical *natural boundary condition* derived from the Rayleigh variational principle in subsection 2.2.1. It is also known as the *Navier slip condition*.

Definition 2.3.1 (Natural boundary condition). *Let $(\text{grad } \mathcal{F})|_{\mathbf{u}} \in \mathfrak{X}_{\text{div}}$ be the gradient of a functional $\mathcal{F}: \mathfrak{X}_{\text{div}} \rightarrow \mathbb{R}$ at $\mathbf{u} \in \mathfrak{X}_{\text{div}}$ as defined in Definition 2.2.3. The **natural boundary condition** associated with $\mathcal{F}$ is the condition on $\mathbf{u} \in \mathfrak{X}_{\text{div}}$ such that (2.9) continues to hold for any variation $\dot{\mathbf{u}} \in \mathfrak{X}_{\text{div}}$ without requiring $\dot{\mathbf{u}}$ to vanish on ∂M .*



Proposition 2.3.1. *The natural boundary condition for $\mathcal{R}$, defined in (2.8), is:*

$$\boxed{(\mathcal{K}\mathbf{u})[\mathbf{t}, \mathbf{n}] = 0} \quad \text{on } \partial M, \quad (2.39)$$

where $\mathbf{t}$ is the unit tangent of the oriented curve ∂M , and $\mathbf{n} = J\mathbf{t}$.

Proof. Section A.5. □

Physically, the condition $(\mathcal{K}\mathbf{u})[\mathbf{t}, \mathbf{n}] = 0$ expresses the absence of friction. The application $(\mathcal{K}\mathbf{u})[\mathbf{n}, \cdot]$ of the tensor $\mathcal{K}\mathbf{u}$ on $\mathbf{n}$ represents the traction force (up to a factor of 2ν) exerted on the boundary geometry ∂M (whose normal is $\mathbf{n}$). Thus, the condition $(\mathcal{K}\mathbf{u})[\mathbf{t}, \mathbf{n}] = 0$ states that this traction force has no tangential component, meaning there is no friction or shear stress from the boundary, allowing the fluid to slip freely along the wall.

Theorem 2.3.1. *Under the natural boundary condition (2.39), the boundary vorticity is:*

$$\boxed{w_{\partial} = 2\kappa_g u_{\partial}} \quad \text{on } \partial M \quad (2.40)$$

where $u_{\partial} := \langle \mathbf{u}, \mathbf{t} \rangle : \partial M \rightarrow \mathbb{R}$ is the tangential velocity component, and κ_g is the geodesic curvature of ∂M .

Proof. Section C.1 □

Example 2.3.1. *The rigid rotating flow in a circular disk serves as an analytical example of (2.40). On the disk of radius R centered at the origin, a rigid rotation with angular velocity a induces a velocity field $(u_x, u_y) = a(-y, x)$, corresponding to a constant vorticity $w = 2a$. The boundary velocity is $u_{\partial} = Ra$ and the boundary curvature is $\kappa_g = \frac{1}{R}$. One can verify $w = 2\kappa_g u_{\partial}$.*

Equation (2.40) can be directly substituted into (2.38). The boundary integral $\oint_{\partial M} \nu h_{i\partial} w_{\partial} ds$ becomes $\oint_{\partial M} 2\nu h_{i\partial} u_{\partial} \kappa_g ds$. We can reorganize the resulting equation geometrically using the following notion.

Definition 2.3.2 (Gauss–Bonnet curvature density). *The Gauss–Bonnet curvature density is a distributional 2-form $\Omega \in \Omega^2(M)$ defined by:*

$$\iint_M f \Omega := \iint_M f K dA + \oint_{\partial M} f \kappa_g ds \quad (2.41)$$

for any smooth function $f \in C^\infty(M)$.

Formally, Ω can be written as $\Omega = (K + \kappa_g \delta_{\partial M}) dA$ where $\delta_{\partial M}$ is the Dirac δ -distribution supported on ∂M . Ω serves as the integrand in the Gauss–Bonnet theorem, satisfying $\iint_M \Omega = 2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M . Thus, Ω generalizes the Gaussian curvature 2-form to surfaces with boundaries, ensuring that integration yields a topological invariant as in the classical Gauss–Bonnet theorem for closed surfaces.

Corollary 2.3.1. *Under the Navier–Stokes equation (2.11) and the free-slip boundary condition (2.39), the harmonic coefficients $(c_1, \dots, c_m)$ evolve according to:*

$$\boxed{\frac{dc_i}{dt} = \iint_M (\mathbf{h}_i \times \mathbf{u}) \omega + 2\nu (\mathbf{h}_i \cdot \mathbf{u}) \Omega.} \quad (2.42)$$

Remark 2.3.1. *While the relationship (2.40) is consistent with the general understanding that boundary curvature and tangential velocity contribute to vorticity generation, a specific*

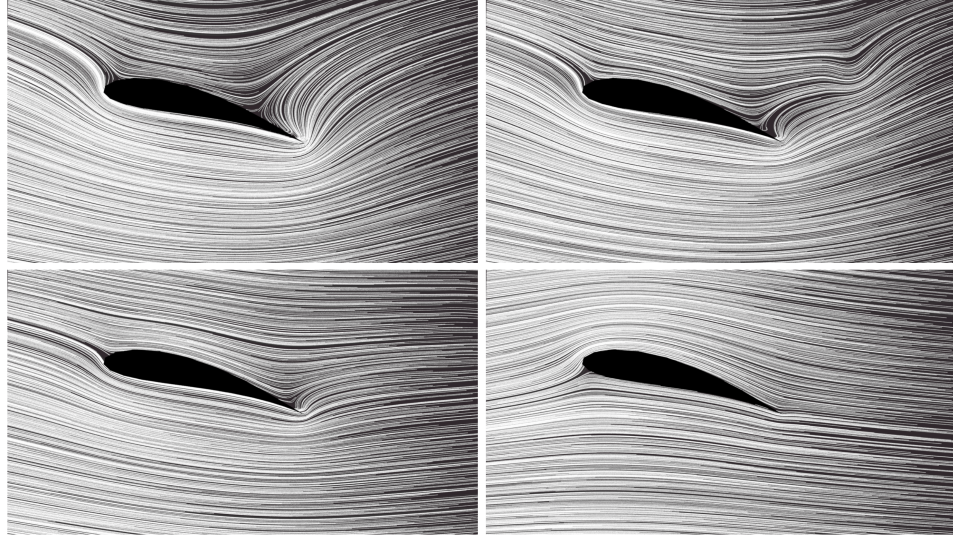


Figure 2.4. The effect of boundary curvature on viscous flow around an obstacle with free-slip boundary condition. The flow around an airfoil with free-slip boundary exhibits nontrivial evolution in its harmonic component. In particular, circulation around the airfoil increases to satisfy the *Kutta condition*—ensuring that the flow does not wrap about the sharp trailing edge. The simulation is performed using Algorithm 1, which solves (2.45), on a cylindrical surface with an embedded airfoil-shaped region removed.

*formulation equating boundary vorticity to the product of geodesic curvature and tangential velocity is not commonly found in standard references. One appearance of (2.40) can be found in [58]. We speculate that the result of Corollary 2.3.1 may lead to a mathematically rigorous foundation for the **Kutta condition**, which argues that steady flows around an airfoil are typically accompanied by a circulation such that the tangential velocity vanishes near the airfoil's tail, where $\kappa_g = \infty$. See Figure 2.4.*

In fact, the boundary condition (2.40) can be elegantly absorbed into the vorticity equation using the Gauss–Bonnet curvature density Ω . Note that the Gauss–Bonnet curvature K^Ω , defined by

$$\Omega = (K + \kappa_g \delta_{\partial M}) dA =: K^\Omega dA, \quad (2.43)$$

exhibits a singular curvature sheet along the boundary, as discussed in Remark 2.2.3.

Theorem 2.2.3 shows that if we replace K by K^Ω in (2.28), a jump $[w]_{\partial M} = 2\kappa_g u_\partial$ arises naturally. To exploit this, we set the Dirichlet boundary condition $w|_{\partial M} = 0$. The vorticity just inside the boundary then matches $2\kappa_g u_\partial$, consistent with (2.40).⁴

Corollary 2.3.2. *The vorticity equation (2.28) with the free-slip condition (2.40) is equivalent to the vorticity equation with using Gauss–Bonnet curvature $K^\Omega = K + \kappa_g \delta_{\partial M}$ with the zero Dirichlet boundary condition:*

$$\begin{cases} \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = \nu \Delta w + 2\nu \operatorname{curl}(K^\Omega \mathbf{u}), \\ w|_{\partial M} = 0 \end{cases} \quad (2.44)$$

That is, an equivalent formulation of the Navier–Stokes equation (2.11) with the free-slip boundary condition (2.39) is the evolution of $(w, c_1, \dots, c_m)$ governed by

$$\begin{cases} \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = \nu \Delta w + 2\nu \operatorname{curl}(K^\Omega \mathbf{u}), \\ \frac{dc_i}{dt} = \iint_M (\mathbf{h}_i \times \mathbf{u}) \omega + 2\nu (\mathbf{h}_i \cdot \mathbf{u}) \Omega, \\ w|_{\partial M} = 0. \end{cases} \quad (2.45)$$

2.3.2 Navier friction boundary

The natural boundary condition (2.39) imposes zero shear stress at the boundary. If instead there is friction from the wall, the boundary condition becomes:

$$\boxed{2\nu(\mathcal{K}\mathbf{u}) \llbracket \mathbf{t}, \mathbf{n} \rrbracket = \alpha u_\partial} \quad \text{on } \partial M \quad (2.46)$$

where $\alpha > 0$ is the friction coefficient.

⁴Here, the jump $[w]_{\partial M}$ of vorticity is evaluated with the vorticity on the outer side of ∂M set to zero, as prescribed by the Dirichlet condition. This is because the singular curvature sheet $\kappa_g \delta_{\partial M}$ is supported slightly inside the domain, which is consistent with the discrete setting described in (2.62) in Section 2.5.1, where the curvature sheet is supported along the triangle strip adjacent to the boundary.

Theorem 2.3.2. *Under the friction condition (2.46), the boundary vorticity is:*

$$\boxed{w_{\partial} = (2\kappa_g - \frac{\alpha}{2\nu})u_{\partial}} \quad \text{on } \partial M, \quad (2.47)$$

where u_{∂} is the tangential velocity component, and κ_g is the geodesic curvature of ∂M .

Proof. Section C.2 □

Equation (2.47) is consistent with the general understanding that for large relative friction ($\alpha/\nu \gg 0$), vorticity generation is approximately proportional to the negative of the tangential velocity [65, 22]. Equation (2.47) precisely specifies that this proportionality constant is the relative friction coefficient offset by the geodesic curvature of the boundary [58].

Similar to the treatment of the free-slip condition using the Gauss–Bonnet curvature (Definition 2.3.2), we can incorporate the condition (2.47) using a modified Gauss–Bonnet curvature. Specifically, we shift the geodesic curvature by the factor $\frac{\alpha}{4\nu}$.

Definition 2.3.3 (Gauss–Bonnet curvature with friction). *The **Gauss–Bonnet curvature density with friction** $\frac{\alpha}{4\nu}$ is a distributional 2-form $\Omega^{\alpha/(4\nu)}$ defined by*

$$\iint_M f \Omega^{\alpha/(4\nu)} := \iint_M f K dA + \oint_{\partial M} f (\kappa_g - \frac{\alpha}{4\nu}) ds \quad (2.48)$$

for any smooth function $f \in C^\infty(M)$. The associated curvature function $K^{\Omega, \alpha/\nu}$ is defined by

$$K^{\Omega, \alpha/(4\nu)} := K + (\kappa_g - \frac{\alpha}{4\nu})\delta_{\partial M}, \quad \Omega^{\alpha/(4\nu)} = K^{\Omega, \alpha/(4\nu)} dA. \quad (2.49)$$

Corollary 2.3.3. *An equivalent formulation of the Navier–Stokes equation (2.11) with the*

Navier friction boundary condition (2.46) is the evolution of $(w, c_1, \dots, c_m)$ governed by

$$\begin{cases} \frac{\partial w}{\partial t} + \mathbf{u} \cdot \nabla w = \nu \Delta w + 2\nu \operatorname{curl}(K^{\Omega, \alpha/(4\nu)} \mathbf{u}), \\ \frac{dc_i}{dt} = \iint_M (\mathbf{h}_i \times \mathbf{u}) \omega + 2\nu (\mathbf{h}_i \cdot \mathbf{u}) \Omega^{\alpha/(4\nu)}, \\ w|_{\partial M} = 0. \end{cases} \quad (2.50)$$

2.3.3 No-slip boundary

The no-slip boundary condition is defined by

$$\boxed{u_{\partial} = 0} \quad \text{on } \partial M, \quad (2.51)$$

where u_{∂} denotes the tangential velocity at ∂M . The condition can be generalized to

$$u_{\partial} = \bar{u}_{\partial} \quad \text{on } \partial M, \quad (2.52)$$

where $\bar{u}_{\partial}$ is a prescribed tangential velocity that can be nonzero. Condition (2.51) can be viewed as the limiting case of the Navier friction condition (2.46) as $\alpha/\nu \rightarrow \infty$. Unlike the free-slip condition (2.40) or the friction condition (2.47), the no-slip condition (2.51) does not directly prescribe the the boundary vorticity w_{∂} . Instead, the boundary vorticity is determined implicitly through the system's dynamics. In the computational setting, the boundary vorticity is characterized as follows:

The boundary vorticity $w_{\partial}: \partial M \rightarrow \mathbb{R}$ is chosen so that, after one time step of simulating the full system (2.38) using this boundary vorticity, the reconstructed velocity (2.31) satisfies $u_{\partial} = 0$.

This implicit step is implemented by a fixed point iteration as detailed in Section 2.5.3.

Figure 2.5 shows an example of (2.52) imposed on a surface with moving boundary.



Figure 2.5. On the Schwarz P surface, an initially stationary fluid ($w = 0$, $c = \mathbf{0}$) is set into motion by a rotating boundary with prescribed velocity $\bar{u}_\partial = 2$. The no-slip condition (2.52) is enforced using Algorithm 2 with time step $\Delta t = 0.01$. Each boundary circle has radius 0.4, and the viscosity is set to $\nu = 0.01$.

Discussion

Incorporating the no-slip boundary condition into the vorticity formulation is not straightforward. There is a long history of research on this topic [46, 95]. Here, we briefly review key developments and clarify certain points, particularly revising some statements where harmonic components were typically neglected in prior work.

The no-through and no-slip conditions translate into $\psi = \frac{\partial \psi}{\partial n} = 0$ for the streamfunction ψ at the boundary.⁵ Early approaches imposed $\Delta \psi = w$ together with $\psi = \frac{\partial \psi}{\partial n} = 0$ at the boundary. Thom [115], Roache [99], and others (see the comprehensive review by E and Liu [35]) set $w_\partial = \Delta \psi|_{\partial M}$. However, these conditions were later shown to be overdetermined. To address this, E and Liu [35] suggested imposing $w_\partial = \Delta \psi|_{\partial M}$ in an explicit time-splitting scheme, noting that it produces roughly the correct amount of vortex sheet.

The problem of overdetermined boundary condition $\psi = \frac{\partial \psi}{\partial n} = 0$ was pointed out by [91, 92]. This overdetermination constraints the vorticity field w to a smaller admissible subspace. Explicitly, it requires that w be L^2 -orthogonal to all harmonic functions on M . However, this characterization did not consider harmonic components and thus

⁵The condition $\psi = \frac{\partial \psi}{\partial n} = 0$ is no longer strictly true in the presence of harmonic components. With harmonic components included, the condition becomes $\psi = \frac{\partial \psi}{\partial n} + \sum_{i=1}^m c_i h_{i;\partial} = 0$.

fails in non-simply-connected domains. Here, we provide a revised version of the Quartapelle–Valz–Gris condition:

Theorem 2.3.3 (Revised Quartapelle–Valz–Gris condition). *For each $f: \partial M \rightarrow \mathbb{R}$, let $\phi_f: M \rightarrow \mathbb{R}$ denote its harmonic extension to the interior, defined by the Dirichlet problem $\Delta \phi_f = 0$ with $\phi_f|_{\partial M} = f$. Under the Navier–Stokes equations with the no-slip condition, the vorticity w and the harmonic coefficients $(c_1, \dots, c_m)$ satisfy*

$$\iint_M \phi_f w \, dA + \sum_{i=1}^m c_i \oint_{\partial M} f h_{i\partial} \, ds = 0 \quad (2.53)$$

for all $f: \partial M \rightarrow \mathbb{R}$.

Proof. See Section C.3. □

A modern understanding of the vorticity and the no-slip condition emphasizes that vortices are generated at the boundary to maintain the no-slip condition. Lighthill [65] proposed vorticity generation proportional to the negative of the slip velocity. Chorin [21, 22, 23] developed vortex sheet boundary layer models aligned with Lighthill’s ideas. Anderson [4] showed that the vorticity flux from the boundary can be derived from the Quartapelle–Valz–Gris condition. See also [62, 59]. Our implicit solve for the no-slip condition follows this general approach.

Cottet [24, 25] introduces a different method, recently adopted by [124] in the computer graphics community. Rather than using a streamfunction, Cottet reconstructs velocity from vorticity via the equation $(d\delta + \delta d)\mathbf{u}^\flat = \delta\omega$ with $\mathbf{u} = \mathbf{0}$ on the boundary. This equation does not directly imply the divergence-free condition $\delta\mathbf{u}^\flat = 0$. Cottet additionally suggests imposing $\omega = d\mathbf{u}^\flat$ at the boundary in the convection-diffusion equation for vorticity. This is implemented as a Neumann condition $\frac{\partial \omega}{\partial \mathbf{n}} = \frac{\partial}{\partial \mathbf{n}}(\text{curl } \mathbf{u}) - \frac{1}{|\partial M|} \oint_{\partial M} \frac{\partial}{\partial \mathbf{n}}(\text{curl } \mathbf{u})$, where the mean-removal term is to enforce $\iint \omega = 0$ at all time, based on the circulation theorem (or as a special case of the classical Quartapelle–Valz–

Gris condition).

Wang et al. [124] recently applied Cottet’s boundary treatment in graphics. They use it for the velocity reconstruction via $-\Delta \mathbf{u} = \text{grad } w$ (in 2D) with $\mathbf{u} = \mathbf{0}$ at the boundary. In this Poisson problem adjacent to the boundary, the value of w_∂ is required; they set $w_\partial = \text{curl } \mathbf{u}$ by back-referencing the velocity, forming a modified Poisson equation [124, § 4.1–4.2]. However, this system is degenerate when the domain is non-simply-connected [124, § 7]. Although their solver appears to select solutions that yield qualitatively reasonable dynamics [124, Fig. 11], the scheme enforces only $\mathbf{u} \cdot \mathbf{n} = 0$ and the tangential part of the velocity is not fully constrained.

2.4 Non-orientable surfaces

The core theory presented in the preceding sections assumes that the surface M is orientable. The 90° rotation operator $\mathcal{J}$ and the Hodge stars $\star$ both depend on the orientation of M . In particular, the vorticity function, defined as the Hodge star of the vorticity 2-form, is not globally well-defined on non-orientable surfaces, as its sign flips when traversing, e.g., a Möbius strip. Nevertheless, fluids can flow on both orientable and non-orientable surfaces. How, then, can we apply the vorticity formulation on a non-orientable surface?

In this section, we extend our theory to fluids on non-orientable surfaces by introducing a useful technique: doubling the domain M to obtain a closed, orientable surface $\tilde{M}$. We show that solving the Navier–Stokes equations on $\tilde{M}$ is equivalent to solving them on M . This approach enables fluid simulations on a non-orientable surface M by solving Navier–Stokes equation on its orientable double cover $\tilde{M}$. The formulation also provides a new perspective on the natural boundary condition (2.40) (Section 2.4.3).

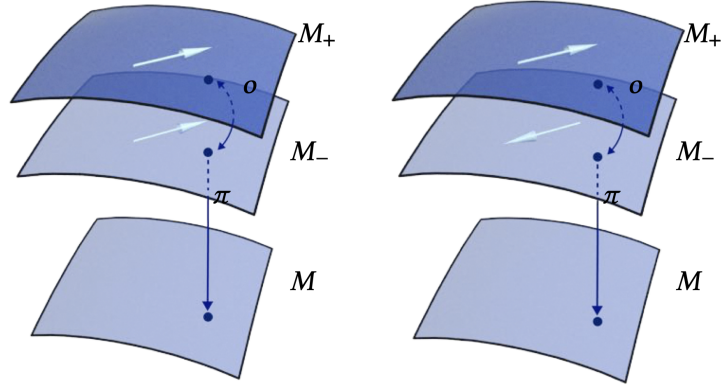


Figure 2.6. Even (left) and odd (right) vector fields on $\tilde{M}$

2.4.1 Double cover

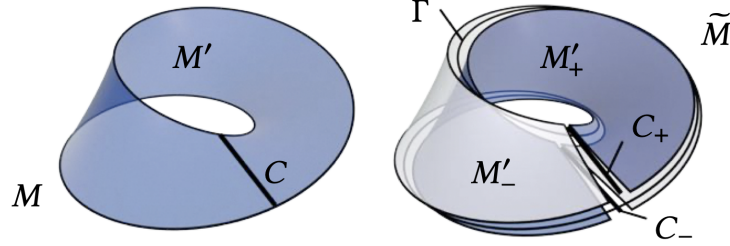
The following construction of *double cover* turns any surface M with boundary and any closed non-orientable surface M into a *closed orientable* surface $\tilde{M}$.

Case 1 M is *orientable and has boundary*. The double cover $\tilde{M}$ is constructed by taking two copies of M and gluing their corresponding boundary points together. More precisely, let $M_+, M_- \simeq M^\circ$ be two copies of the interior of M , and let $\Gamma \simeq \partial M$ be a copy of the boundary. Define $\tilde{M} := M_+ \sqcup \Gamma \sqcup M_-$, and let $\pi: \tilde{M} \rightarrow M$ be the projection that maps each $x \in \tilde{M}$ to the point in M from which it was copied. The topology of $\tilde{M}$ is defined so that π is continuous.

Case 2 M is *non-orientable*. Let $\tilde{M}$ be the standard oriented double cover of M with the boundary points glued together. A concrete construction goes as follows. Choose a set of curves $C \subset M^\circ$ as cuts in the interior M° of M , so that $M' := M^\circ \setminus C$ is orientable (i.e., M' is a maximal oriented subdomain of M°).⁶ Let $M'_+, M'_- \simeq M'$ be two copies of M' , let $C_+, C_- \simeq C$ be two copies of C , and let $\Gamma \simeq \partial M$ be a copy of the boundary (which may be empty). Define $\tilde{M} := M'_+ \sqcup M'_- \sqcup C_+ \sqcup C_- \sqcup \Gamma$,

⁶In practice, for a mesh M , one may use a spanning tree to orient the faces of M , with C given by the edges where adjacent faces remain inconsistently oriented.

and let $\pi: \tilde{M} \rightarrow M$ be the projection that sends each $x \in \tilde{M}$ to the point in M from which it originates.



The topology of $\tilde{M}$ is defined so that it is orientable—meaning, in particular, that C_+ is adjacent to M'_+ on one side and M'_- on the other—and such that π is continuous. The resulting object $(\tilde{M}, \pi)$ is independent of the choice of the cuts C .

Note that $\tilde{M}$ is always an orientable closed surface by construction. If M is orientable with genus g and b boundary components, then $\text{genus}(\tilde{M}) = 2g + \max(b - 1, 0)$. If M is non-orientable and is the connected sum of k real projective planes with b disks removed, then $\text{genus}(\tilde{M}) = k - 1 + b$.

The double cover $(\tilde{M}, \pi)$ is naturally equipped with an involutive map $o: \tilde{M} \rightarrow \tilde{M}$ that swaps the corresponding points in M_+ and M_- for orientable M , and swaps the corresponding points in M'_+, M'_- and C_+, C_- for non-orientable M . Explicitly, o is defined by $\pi \circ o = \pi$, and $o(x) \neq x$ for $x \notin \Gamma$. The map o is an involution, i.e., $o^2 = \text{id}_{\tilde{M}}$, and Γ is the set of fixed points of o .

Both π and o are isometries.

Curvature of $\tilde{M}$

Since M_+ and M_- (resp. M'_+ and M'_-) are isometric to M° (resp. M'), the Gaussian curvature on $\tilde{M} \setminus \Gamma$ is given by $K \circ \pi$. We may abbreviate $K \circ \pi$ as K since they have the same values. However, the Gaussian curvature $\tilde{K}$ of $\tilde{M}$ is not simply K everywhere. The interface Γ forms a sharp ridge, with concentrated Dirac- δ Gaussian curvature.

Precisely:

$$\tilde{K} = K + 2\kappa_g \delta_\Gamma. \quad (2.54)$$

Even and odd fields

The map $o: \tilde{M} \rightarrow \tilde{M}$ induces a pullback operator o^* that acts on fields. These fields can be scalar functions $\Omega^0(\tilde{M})$, vector fields $\mathfrak{X}(\tilde{M})$, or differential forms $\Omega^k(\tilde{M})$. The pullback o^* is linear and involutive (since o is an involution). Thus, the eigenvalues of o^* are ± 1 . We refer to elements of the $+1$ -eigenspace as *even field*, and to elements of the -1 -eigenspace as *odd field*. That is, an even field f satisfies $o^*f = f$, and an odd field satisfies $o^*f = -f$.⁷

Every field f admits a unique decomposition into even and odd parts: $f = \frac{f+o^*f}{2} + \frac{f-o^*f}{2}$.

Even functions take the same value at opposite points, while odd functions take values of equal magnitude but opposite sign. In particular, odd functions vanish on Γ .

Even vector fields v satisfy $do(v) = v$; odd ones satisfy $do(v) = -v$. See Figure 2.6. Divergence-free even vector fields must be tangent to Γ . In particular, $\mathfrak{X}_{\text{div}}(M) \cong \mathfrak{X}_{\text{div,even}}(\tilde{M})$.

The operators $\sharp, \flat, d, \text{grad}, \text{div}$ preserve the evenness and oddness. The Hodge star $\star$ and the 90° rotation operator J switch the even/odd parity. Wedge product and inner product respect the parity: $\text{even} \cdot \text{even} = \text{odd} \cdot \text{odd} = \text{even}$, and $\text{even} \cdot \text{odd} = \text{odd}$.

Hodge decomposition

Since the exterior calculus operators respect parity, the Hodge decomposition respects the even/odd splitting. Note that 1-forms on M satisfying the co-Dirichlet

⁷Odd differential forms are sometimes referred to as *pseudoforms*, *twisted forms*, or forms of *odd type*, as introduced in de Rham's work [30].



Figure 2.7. Among the two-dimensional space of harmonic fields on the torus—the oriented double cover of the Klein bottle—only one dimension, corresponding to the even harmonic field, contributes to the harmonic vector field on the Klein bottle, as shown on the left. The other dimension, spanned by an odd harmonic field, does not descend to the Klein bottle, as illustrated on the right by the visible discontinuity.

(Neumann) boundary condition correspond to $\Omega_{\text{even}}^1(\tilde{M})$, and 1-forms satisfying the Dirichlet boundary condition correspond to $\Omega_{\text{odd}}^1(\tilde{M})$. These admit the Hodge decomposition:

$$\Omega_{\text{even}}^1(\tilde{M}) = d\Omega_{\text{even}}^0(\tilde{M}) \oplus \delta\Omega_{\text{even}}^2(\tilde{M}) \oplus \mathcal{H}_{\text{even}}^1(\tilde{M}), \quad (2.55)$$

$$\Omega_{\text{odd}}^1(\tilde{M}) = d\Omega_{\text{odd}}^0(\tilde{M}) \oplus \delta\Omega_{\text{odd}}^2(\tilde{M}) \oplus \mathcal{H}_{\text{odd}}^1(\tilde{M}). \quad (2.56)$$

The space of velocity 1-forms $\mathfrak{X}_{\text{div}}(M)^b$ corresponds to $\delta\Omega_{\text{even}}^2(\tilde{M}) \oplus \mathcal{H}_{\text{even}}^1(\tilde{M})$. Thus, harmonic vector fields in $\mathfrak{X}_{\text{div}}(M)$ correspond to even harmonic 1-forms, while stream-functions are odd functions (or equivalently, even 2-forms).

If M is non-orientable, its double cover $\tilde{M}$ has $2 \cdot \text{genus}(\tilde{M})$ linearly independent harmonic 1-forms, half of which are even. For example, the Klein bottle is double-covered by a torus. The torus has a 2D space of harmonic fields—one even and one odd—so only one harmonic field contributes to the harmonic velocity on the Klein bottle. See Figure 2.7.

Field	Symbol	Type
Velocity	$\mathbf{u}$	even div-free vector field
Velocity 1-form	$\mathbf{u}^\flat$	even 1-form
Vorticity function	w	odd function
Vorticity 2-form	ω	even 2-form
Streamfunction	ψ	odd function
Harmonic fields	$\mathbf{h}_i$	even harmonic vector field
Gaussian curvature	$\tilde{K}$	even function
Gaussian curvature density	$\tilde{\Omega}$	even density or odd 2-form

Table 2.1. Type of fields on the double cover $\tilde{M}$.

2.4.2 Navier–Stokes equations on the double cover

Each quantity in the Navier–Stokes system on M lifts to either an even or odd field on $\tilde{M}$, as summarized in Table 2.1. If the initial data respect this parity structure, then the evolution under the Navier–Stokes equations preserves it.

This means our Navier–Stokes formulation for orientable surfaces directly extends to non-orientable ones, by simulating the dynamics on the orientable double cover using initial data with the parity structure described in Table 2.1.

2.4.3 Relation to boundary conditions

As shown in (2.54), the Gaussian curvature of $\tilde{M}$ contains a singular curvature sheet along Γ . By Theorem 2.2.3, the vorticity function w exhibits a jump across Γ :

$$[w]_\Gamma = 4\kappa_g u_\partial. \quad (2.57)$$

Since w is an odd function on $\tilde{M}$, it vanishes at Γ , and has opposite limiting values from either side: $2\kappa_g u_\partial$ from M^+ , and $-2\kappa_g u_\partial$ from M^- . Thus, we recover the natural (free-slip) boundary condition (2.40) on M automatically.

To simulate Navier friction or no-slip boundary conditions on M using $\tilde{M}$, one cuts Γ open, thereby turning $\tilde{M}$ into a surface with boundary, and imposes the desired

boundary condition on $\partial\tilde{M}$.

2.5 Discretization

Our numerical method for simulating (2.38) is based on the discretization following [8] and [127].

For spatial discretization, the surface M is represented by a triangle mesh (P, E, F) , where P is the set of vertices, E the set of edges, and F the set of faces. The vorticity function w and the streamfunction ψ are discretized at vertices, while the vector fields $\mathbf{u}$, $\mathbf{h}_i$ are discretized on faces. The discrete Gauss–Bonnet curvature is also assigned to faces. Details of the discrete operators and the face-based discrete Gauss–Bonnet curvature are provided in Section 2.5.1.

For time integration, we employ an *implicit-explicit (IMEX) scheme*. Specifically, we split the system into a *stiff part* and a *non-stiff part*. The stiff part consists of the viscous term $\nu(\Delta w + 2 \operatorname{curl}(K\mathbf{u}))$ in the vorticity equation, while the remaining terms in the vorticity equation and the evolution equation of the harmonic components are classified as non-stiff. The non-stiff part is integrated with an explicit 4th-order Runge–Kutta (RK4) method [127], alternating with an implicit backward Euler method applied to the stiff viscous term. The full procedure is described in Section 2.5.2.

For simplicity, we first present the method for system with the free-slip boundary condition; the incorporation of the friction and no-slip boundary conditions is described separately in Section 2.5.3.

2.5.1 Discrete surface and its operators

Discrete differential operators

A discrete vector field $\mathbf{u} \in \Gamma_F(TM)$ is defined as a piecewise constant vector $\mathbf{u}_f \in T_f M$ assigned on each triangle $f \in F$. The directional derivative operator associated

with $\mathbf{u}$, denoted as $[\nabla_{\mathbf{u}}]: \mathbb{R}^{|P|} \rightarrow \mathbb{R}^{|P|}$, acts on a discrete function $f = (f_p)_{p \in P} \in \mathbb{R}^{|P|}$ as

$$([\nabla_{\mathbf{u}}]f)_p := \frac{1}{\sum_{f \succ p} A_f} \sum_{f \succ p} A_f \langle \mathbf{u}_f, (\text{grad } f)_f \rangle \quad (2.58)$$

where A_f is the area of triangle f , and $\text{grad}: \mathbb{R}^{|P|} \rightarrow \Gamma_F(TM)$ computes the piecewise constant gradient of the linear interpolation of the discrete function f .

The divergence operator $\text{div}: \Gamma_F(TM) \rightarrow \mathbb{R}^{|P|}$ is defined as the negative L^2 -adjoint of the gradient, $\text{div} = -\text{grad}^*$. Explicitly, $\text{div} = -A_P^{-1} \text{grad}^T A_F$ where A_F is the diagonal matrix of triangle areas, and A_P is the diagonal matrix of the vertex areas.

The scalar curl operator is defined as the composition of the divergence with the facewise 90° counterclockwise rotation: $J_F \mathbf{u} = (J_f \mathbf{u}_f)_{f \in F}$. That is, $\text{curl} = -\text{div} \circ J_F: \Gamma_F(TM) \rightarrow \mathbb{R}^{|P|}$.

The Laplacian $\Delta = \text{div} \circ \text{grad}: \mathbb{R}^{|P|} \rightarrow \mathbb{R}^{|P|}$ corresponds to the cotangent Laplacian on mesh M . When assembled in matrix form, it takes the form $\Delta = -A_P^{-1} L \in \mathbb{R}^{|P| \times |P|}$, where $L = \text{grad}^T A_F \text{grad}$ is the cotangent stiffness matrix.

Interior and boundary vertices

We decompose the vertex set into the interior and boundary parts: $P = I \sqcup B$, where I and B denote the interior and boundary vertex sets. If M is a closed surface, then $P = I$ and $B = \emptyset$.

Discrete Gauss curvature

We discretize Gaussian curvature at faces as the total *angle defect* divided by the triangle area, defined as follows.

For each vertex p incident to face f , let $\angle_{p,f}$ denote the interior angle of the triangle

f at vertex p . Define the *rescaled interior angle* as

$$\tilde{\angle}_{p,f} := \begin{cases} \frac{2\pi}{\sum_{f' \succ p} \angle_{p,f'}} \angle_{p,f} & \text{for } p \in I \\ \angle_{p,f} & \text{for } p \in B. \end{cases} \quad (2.59)$$

This angle is rescaled in the sense that the total angle around an interior vertex is $\sum_{f \succ p} \tilde{\angle}_{p,f} = 2\pi$. Geometrically, $\tilde{\angle}_{p,f}$ is the geodesic corner angle obtained by smoothing the vertex cone and measuring angles in the tangent plane at the vertex. This modification of angle, however, no longer satisfies the total interior angle of a triangle being π , and thereby renders a total angle defect in each triangle. For each triangle $f \in F$, define the *discrete Gaussian curvature* $K \in \mathbb{R}^{|F|}$ as

$$K_f := \frac{1}{A_f} (\sum_{p \prec f} \tilde{\angle}_{p,f} - \pi). \quad (2.60)$$

On a closed surface, $(K_f)_{f \in F}$ satisfies the discrete Gauss–Bonnet theorem $\sum_f A_f K_f = 2\pi\chi(M)$, where $\chi(M) = |P| - |E| + |F|$ is the Euler characteristic.

To define the *discrete Gauss–Bonnet curvature* K_f^Ω (cf. (2.3.2)) we must incorporate the geodesic curvature into K_f . We do so by introducing another rescaled angle similar to (2.59) but this time also rescales the boundary angles:

$$\tilde{\angle}_{p,f}^\Omega := \begin{cases} \frac{2\pi}{\sum_{f' \succ p} \angle_{p,f'}} \angle_{p,f} & \text{for } p \in I \\ \frac{\pi}{\sum_{f' \succ p} \angle_{p,f'}} \angle_{p,f} & \text{for } p \in B. \end{cases} \quad (2.61)$$

Define the discrete Gauss–Bonnet curvature at face f as

$$K_f^\Omega := \frac{1}{A_f} (\sum_{p \prec f} \tilde{\angle}_{p,f}^\Omega - \pi). \quad (2.62)$$

When the surface has no boundary, $K_f^\Omega = K_f$. In general, these curvatures satisfy the

discrete Gauss–Bonnet theorem with boundary: $\sum_f A_f K_f^\Omega = 2\pi\chi(M)$.

2.5.2 Main algorithm with the free-slip condition

We now discretize the system (2.38) governing the vorticity field $w \in \mathbb{R}^{|P|}$ and the coefficient tuple $\mathbf{c} = (c_i)_{i=1}^m \in \mathbb{R}^m$. In this section, we assume the free-slip boundary condition (Section 2.3.1). Specifically, we discretize system (2.45). The modification into other boundary conditions is detailed in Section 2.5.3.

Since the vorticity vanishes on the boundary following the formulation of (2.45), we have $w_B = 0$. Thus, the fluid state is reduced to an *interior* vorticity field $w_I \in \mathbb{R}^{|I|}$ and $\mathbf{c} \in \mathbb{R}^m$. Given a precomputed L^2 -orthonormal harmonic basis $(\mathbf{h}_i)_{i=1}^m$ with $\mathbf{h}_i \in \Gamma_F(TM)$, the corresponding velocity field $\mathbf{u} \in \Gamma_F(TM)$ is obtained by solving the Poisson problem (2.32) followed by the reconstruction step (2.31), using the discrete operators introduced in subsection 2.5.1. We denote this process as $\mathbf{u} = \text{VELOCITY}(w_I, \mathbf{c}; \mathbf{h})$:

$$\text{VELOCITY}(\cdot, \cdot; \mathbf{h}) : \mathbb{R}^{|I|} \times \mathbb{R}^m \rightarrow \Gamma_F(TM) \quad (2.63a)$$

$$\text{VELOCITY}(w_I, \mathbf{c}; \mathbf{h}) := -J \text{grad } \psi + \sum_{i=1}^m c_i \mathbf{h}_i \quad (2.63b)$$

$$\text{where } L_{II} \psi_I = (A_P)_{II} w_I, \quad \psi_B \equiv 0. \quad (2.63c)$$

Using this velocity and the discrete operators and curvature defined in Section 2.5.1, the system (2.45) translates into the following semi-discrete form:

$$\frac{d}{dt} w_I = \overbrace{-[\nabla_{\mathbf{u}}]_{(I,:)} w}^{\text{ADVECT}(\mathbf{u}, w)} + \overbrace{\nu \Delta_{(I,:)} w + 2\nu \text{curl}_{(I,:)}((K_f^\Omega \mathbf{u}_f)_{f \in F})}^{\text{DIFFUSE}(\mathbf{u}, w)}, \quad (2.64a)$$

$$\frac{d}{dt} c_i = \underbrace{\sum_{f \in F} A_f (-w_f \langle J_f \mathbf{u}_f, \mathbf{h}_{i,f} \rangle + 2\nu K_f^\Omega \langle \mathbf{u}_f, \mathbf{h}_{i,f} \rangle)}_{\text{HARMONICRATE}(\mathbf{u}, w)}, \quad (2.64b)$$

where $w_f = \frac{1}{3} \sum_{p \prec f} w_p$ is the vorticity field interpolated to the face, and w_B is kept zero.

We use an IMEX scheme (Algorithm 1) to integrate (2.64) in time, treating the nonstiff terms—namely $\text{ADVECT}(\mathbf{u}, w)$ in (2.64a) and $\text{HARMONICRATE}(\mathbf{u}, w)$ of (2.64b) (cf. EVALRHS in Algorithm 1)—with an explicit fourth-order Runge–Kutta method (cf. RK4STEP), and the stiff term $\text{DIFFUSE}(\mathbf{u}, w)$ with an implicit Euler method: given w_1^{old} at the previous time step, solve the the vorticity w_1^{new} at the end of the time interval satisfying

$$w_1^{\text{new}} - w_1^{\text{old}} = \Delta t \text{DIFFUSE}(\mathbf{u}^{\text{new}}, w^{\text{new}}). \quad (2.65)$$

This is equivalent to

$$(A_P + \nu \Delta t L)_{\parallel} w_1^{\text{new}} = (A_P)_{\parallel} (w_1^{\text{old}} + 2\nu \Delta t \text{curl}_{(l,:)}(K^{\Omega} \mathbf{u}^{\text{new}})) \quad (2.66)$$

which can be solved by a fixed-point iteration (cf. FIXEDPOINT). This fixed-point iteration typically converges in 2–3 iterations with tolerance $\varepsilon = 10^{-5}$.

Algorithm 1. IMEX Scheme for (2.45)

```
1:  $\mathbf{h} = (\mathbf{h}_i)_{i=1}^m \leftarrow$  An orthonormal basis for  $\mathfrak{H}$ ;  
2:  $w = (w_p)_{p \in P} \leftarrow$  Initialize vorticity;  $w_B \leftarrow 0$ ;  
3:  $\mathbf{c} = (c_i)_{i=1}^m \leftarrow$  Initialize harmonic coefficients;  
4:  $\Delta t > 0 \leftarrow$  Set time step;  
5:  $\text{tol} > 0 \leftarrow$  Set fixed-point iteration tolerance;  
6: for each frame do  
7:    $\begin{bmatrix} w_l \\ \mathbf{c} \end{bmatrix} \leftarrow \text{RK4STEP}(\text{EVALRHS}, \begin{bmatrix} w_l \\ \mathbf{c} \end{bmatrix}, \Delta t);$  ▷ explicit step  
8:    $w_l \leftarrow \text{FIXEDPOINT}(w_l, \mathbf{c}, \Delta t, \text{tol});$  ▷ implicit step  
9:   export VELOCITY  $(w_l, \mathbf{c}; \mathbf{h});$   
10: function EVALRHS( $w_l, \mathbf{c}$ )  
11:    $\mathbf{u} \leftarrow \text{VELOCITY}(w_l, \mathbf{c}; \mathbf{h});$  ▷ cf. (2.63)  
12:    $\dot{w}_l \leftarrow \text{ADVECT}(\mathbf{u}, w);$  ▷ cf. (2.64a)  
13:    $\dot{\mathbf{c}} \leftarrow \text{HARMONICRATE}(\mathbf{u}, w);$  ▷ cf. (2.64b)  
14:   return  $(\dot{w}_l, \dot{\mathbf{c}});$   
15: function FIXEDPOINT( $w_l, \mathbf{c}, \Delta t, \text{tol}$ )  
16:    $w_l^{\text{old}} \leftarrow w_l;$   
17:   do ▷ cf. (2.65)  
18:      $\mathbf{u} \leftarrow \text{VELOCITY}(w_l, \mathbf{c}; \mathbf{h});$   
19:      $\tilde{w}_l \leftarrow w_l^{\text{old}} + 2\nu\Delta t \text{curl}_{(l, \cdot)}(K^\Omega \mathbf{u});$   
20:      $\tilde{w}_l \leftarrow (A_P + \nu\Delta t L)_{||}^{-1} (A_P)_{||} \tilde{w}_l;$   
21:      $\varepsilon \leftarrow \|\tilde{w} - w\|; w \leftarrow \tilde{w};$   
22:   while  $\varepsilon > \text{tol}$   
23:   return  $w_l;$   
24: function RK4STEP( $F \in (\mathbb{T} \rightarrow \mathbb{T}), x \in \mathbb{T}, \Delta t \in \mathbb{R}$ )  
25:    $k_1 \leftarrow F(x); k_2 \leftarrow F(x + \Delta t/2 \cdot k_1);$   
26:    $k_3 \leftarrow F(x + \Delta t/2 \cdot k_2); k_4 \leftarrow F(x + \Delta t \cdot k_3);$   
27:   return  $x + \Delta t/6 \cdot (k_1 + 2k_2 + 2k_3 + k_4);$ 
```

2.5.3 Friction and no-slip boundary conditions

Algorithm 1 is a solver for fluids on closed surfaces or surfaces with free-slip boundary. We now describe its modifications into having friction boundary and no-slip boundary.

Navier friction boundary

As in the continuous theory (Section 2.3.2), adding friction amounts to modifying the Gauss–Bonnet curvature at the boundary (see (2.49) and (2.50)).

Continuing the definition of discrete Gauss–Bonnet curvature in Section 2.5.1, we introduce the rescaled angle that incorporate friction (cf. (2.61)):

$$\tilde{Z}_{p,f}^{\Omega,\alpha/(4\nu)} := \frac{\pi}{(\sum_{f \succ p} \angle_{p,f}) + \frac{\alpha}{4\nu} \ell_p} \angle_{p,f} \quad \text{for } p \in B \quad (2.67)$$

where ℓ_p is the averaged boundary edge length at vertex $p \in B$. For interior vertices $p \in I$, there is no modification: $\tilde{Z}_{p,f}^{\Omega,\alpha/(4\nu)} = \tilde{Z}_{p,f}^{\Omega}$. Define the *discrete Gauss–Bonnet curvature with friction* $\frac{\alpha}{4\nu}$:

$$K_f^{\Omega,\alpha/(4\nu)} := \frac{1}{A_f} \left(\sum_{p \prec f} \tilde{Z}_{p,f}^{\Omega,\alpha/(4\nu)} - \pi \right). \quad (2.68)$$

To incorporate the friction boundary, the only algorithmic modification is replacing K^Ω by $K^{\Omega,\alpha/(4\nu)}$ in Algorithm 1.

Geometrically, this modification amounts to overwriting and inflating the value of the total interior angle of the boundary polygon $\sum_{f \succ p} \angle_{p,f}$, and thereby decreasing the exterior angle echoing the expression $(\kappa_g - \frac{\alpha}{4\nu})$ in (2.49). The net result decreases the rescaled angle $\tilde{Z}$ at the boundary, introducing negative Gaussian curvature into the neighboring triangle.

The limiting case of infinity friction $\frac{\alpha}{4\nu} \rightarrow \infty$ corresponds to setting the rescaled

angle to zero:

$$\tilde{Z}_{p,f}^{\Omega,\infty} = 0 \quad \text{for } p \in B. \quad (2.69)$$

No-slip boundary

One way to approximate the no-slip boundary condition is to apply the friction boundary (Section 2.5.3) with infinite friction coefficient, i.e., setting the rescaled angle to zero (2.69).

Alternatively, we impose the condition $u_{\partial} = 0$ by introducing a non-zero boundary vorticity w_B as in the description in Section 2.3.3.

For this latter setting, the system equation is reverted to (2.38) with a nonzero w_B . In particular, the Gaussian curvature K_f^{Ω} is replaced by K_f (given by (2.60)), that is, without the curvature sheet from the boundary. Note that with a non-zero w_B , a boundary contribution $\sum_{e \in \partial M} \nu \langle (\mathbf{h}_i)_{f \succ e}, \mathbf{e}_e \rangle w_e \ell_e$ need to be added in (2.64b). Here, $\mathbf{e}_e$ is the unit edge direction and ℓ_e the edge length of edge e , and $w_e = \sum_{p \prec e} \frac{w_p}{2}$ for $p \in B$ denotes the vorticity field interpolated to the boundary edge.

The boundary vorticity field w_B is chosen to ensure that the boundary velocity

$$(u_{\partial})_e = \langle \mathbf{u}_{f \succ e}, \mathbf{e}_e \rangle \quad (2.70)$$

vanishes at each boundary edge $e \in \partial M$ (cf. subsection 2.3.3). This condition is enforced through a modified fixed-point iteration (Algorithm 2).

Algorithm 2. Fixed-Point Iteration for No-Slip Boundary Condition

```
1: function FIXEDPOINTNOSLIP( $w, \mathbf{c} = \{c_i\}, \Delta t, \text{tol}_w, \text{tol}_u, \beta$ )
2:    $w^{\text{old}} \leftarrow w, \mathbf{c}^{\text{old}} \leftarrow \mathbf{c};$ 
3:   do
4:      $\mathbf{u} \leftarrow \text{VELOCITY}(w, \mathbf{c}; \mathbf{h});$ 
5:      $\tilde{w}_I \leftarrow w_I^{\text{old}} + 2\nu\Delta t(\text{curl}(K\mathbf{u}))_I;$ 
6:      $\tilde{w}_B \leftarrow \tilde{w}_B + \beta(u_\partial)_B \quad \triangleright (u_\partial)_B := (2.70) \text{ averaged to vertex}$ 
7:      $\tilde{w}_I \leftarrow M_{\parallel}^{-1}((A_P\tilde{w})_I - M_{IB}\tilde{w}_B), \quad M := A_P + \nu\Delta tL;$ 
8:      $\tilde{c}_i \leftarrow \tilde{c}_i^{\text{old}} + \Delta t \sum_{e \in \partial M} \nu \langle (\mathbf{h}_i)_{f \succ e}, \mathbf{e}_e \rangle \tilde{w}_{\partial,e} \ell_e;$ 
9:      $\varepsilon_w \leftarrow \|\tilde{w} - w\|; w \leftarrow \tilde{w};$ 
10:     $\varepsilon_c \leftarrow \|\tilde{\mathbf{c}} - \mathbf{c}\|; \mathbf{c} \leftarrow \tilde{\mathbf{c}};$ 
11:    while  $\varepsilon_w > \text{tol}_w$  or  $\varepsilon_c > \text{tol}_c$  or  $\|u_\partial\| > \text{tol}_u$ 
12:  return  $w;$ 
```

The relaxation parameter $\beta > 0$ is chosen such that $\beta\ell_p \approx 1$, where ℓ_p is the averaged edge length at p .

High-friction v.s. no-slip

As we will demonstrate in Figure 2.16, the high-friction boundary (Algorithm 1 with (2.69)) and the no-slip boundary (Algorithm 2) produce similar fluid-dynamical outcomes. However, the high-friction boundary treatment is significantly more efficient. The fixed-point iteration in Algorithm 1 remains robust even under infinite friction, whereas the fixed-point iteration in Algorithm 2 can become unstable at large Reynolds numbers, which can be mitigated by having higher resolution near the boundary.

2.6 Numerical examples

Our computation is implemented in SideFX Houdini. These tests were conducted on a Dell 8940 Desktop equipped with an Intel CPU i9-11900K operating at 3.50 GHz,

64 GB of RAM. For typical meshes—such as the example shown in Figure 2.15 with approximately 22K vertices— under free-slip or friction boundary conditions, the simulation using Algorithm 1 takes about 90 minutes in total (about 5 seconds per frame). For no-slip condition shown in Figure 2.5, which has 16K vertices, the computation takes about 27 minutes (about 6 seconds per frame) using Algorithm 1 and Algorithm 2.

2.6.1 Viscous Laplacian

A common treatment of viscous dissipative in surface vortex dynamics involves only the Laplace–Beltrami operator applied to the vorticity. That is, in this approach the vorticity equation is modeled as $\dot{w} + \nabla_u w = \nu \Delta w$ [36, 126, 37, 8, 121, 111], which is equivalent to replacing the viscous Laplacian (Definition 2.2.9) in the Navier–Stokes equation (2.20) with the Hodge-Laplacian (Definition 2.2.8).

Contrary to this standard approach, we demonstrate that the curvature-dependent term $2\nu \operatorname{curl}(Ku)$ in our vorticity equation (2.28) is crucial for capturing the correct dynamical behavior of fluid flows on curved surfaces.

We compare our method with that of [8], which is based on solving $\dot{w} + \nabla_u w = \nu \Delta w$ without the curvature-dependent term. To isolate the effects of curvature and avoid dependency of boundaries and harmonic components, the comparisons are performed on topological spheres.

In Figure 2.8, the vorticity field is initialized on a round sphere to represent a rigid body rotation. In the absence of shear deformation in a rigid motion, viscous dissipation should not occur, and conservation of angular momentum implies that the fluid should maintain its rotation indefinitely. Our method correctly preserves this rigid motion. In contrast, the model $\dot{w} + \nabla_u w = \nu \Delta w$ fails to do so: the velocity rapidly decays to zero. This behavior is expected, as the Laplace–Beltrami operator on a closed surface has only constant functions in its kernel. Since the vorticity on a closed surface has zero mean, the dissipative term in this model ultimately drives the vorticity to zero.

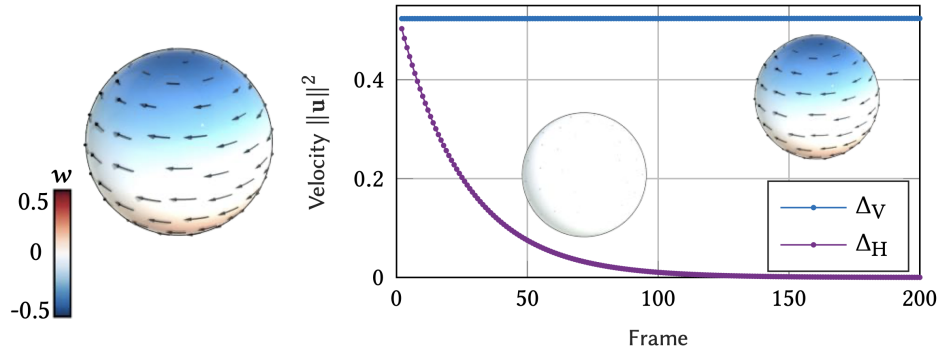


Figure 2.8. Comparison of viscosity modeled using the viscous Laplacian Δ_V (our method) versus the Hodge Laplace Δ_H (previous methods). An initial rigid body rotation is preserved in our model. In contrast, the Hodge Laplacian model—corresponding to the vorticity equation $\dot{w} + \nabla_u w = \nu \Delta w$ —fails to preserve rigid body motion and instead dissipates it.

Figure 2.9 presents another comparison, this time on a surface of revolution with non-uniform curvature. The initial condition is again a rigid body rotation. Our method preserves this motion, while the previous method introduces a non-physical distortion in the flow and eventually decays to rest.

2.6.2 Harmonic dynamics and preservation of Killing modes

We further demonstrate that the evolution for the harmonic component in (2.38) is essential for accurately simulating viscous flows on non-simply-connected surfaces.

In Figure 2.10, we initialize the vorticity field on a torus with $w = -\langle \mathbf{n}, \hat{\mathbf{y}} \rangle$ and $c_1 = c_2 = 0$ on a torus, where $\mathbf{n}$ is the surface normal, and $\hat{\mathbf{y}}$ is the upward-pointing unit vector. We compare the results of our full system (2.38), which includes harmonic evolution, against a version of the system with the harmonic dynamics disabled (i.e., $\dot{c} = 0$). At each time step, we extract both the harmonic component and the Killing component of the velocity field.

When the harmonic evolution is omitted, the Killing fields on the torus fail to be conserved, violating the conservation property stated in Theorem 2.2.2. In contrast, when the harmonic field dynamics are included, the Killing field component is success-

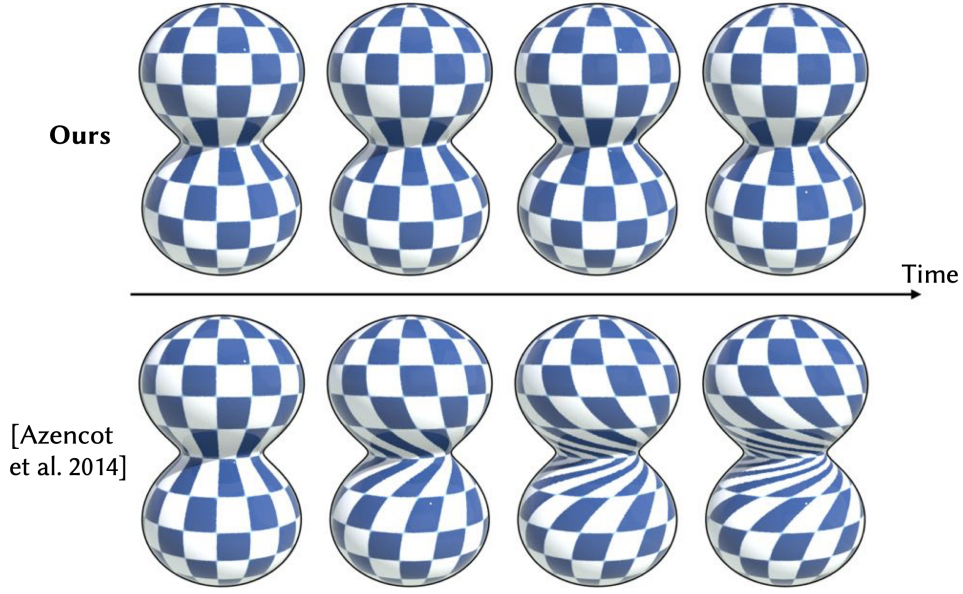


Figure 2.9. Viscous flow on a surface of revolution with non-uniform curvature, initialized with a rigid body rotation. Our method correctly preserve the rigid body motion, and thus the advected textures, while previous vorticity methods using $\dot{w} + \nabla_u w = \nu \Delta w$ produce non-physical distortions. The initial angular velocity is 1 and the viscosity is $\nu = 0.1$. The sequence shows frames 1, 30, 75, 150 with $\Delta t = 0.1$.

fully preserved. Interestingly, we also observe a nontrivial interchange between the harmonic component and the vorticity over time.

Figure 2.11 shows a simulation on an *Enneper surface*, which possesses an intrinsic Killing field that is not the restriction of a rigid body motion in $\mathbb{R}^3$. This surface has two boundaries, where we impose the free-slip boundary condition. Even in this more general setting, our method preserves the Killing mode, in agreement with Theorem 2.2.2. As in Figure 2.10, the initial condition is given by $w = -\langle \mathbf{n}, \hat{\mathbf{y}} \rangle$ and $c_1 = 0$. The flow relaxes to a pure Killing field.

2.6.3 Non-orientable surfaces

We demonstrate fluid simulation on non-orientable surfaces using our algorithm and the double cover technique in Section 2.4.

The *Boy surface* shown in Figure 2.12 has the topology of $\mathbb{RP}^2$, whose oriented

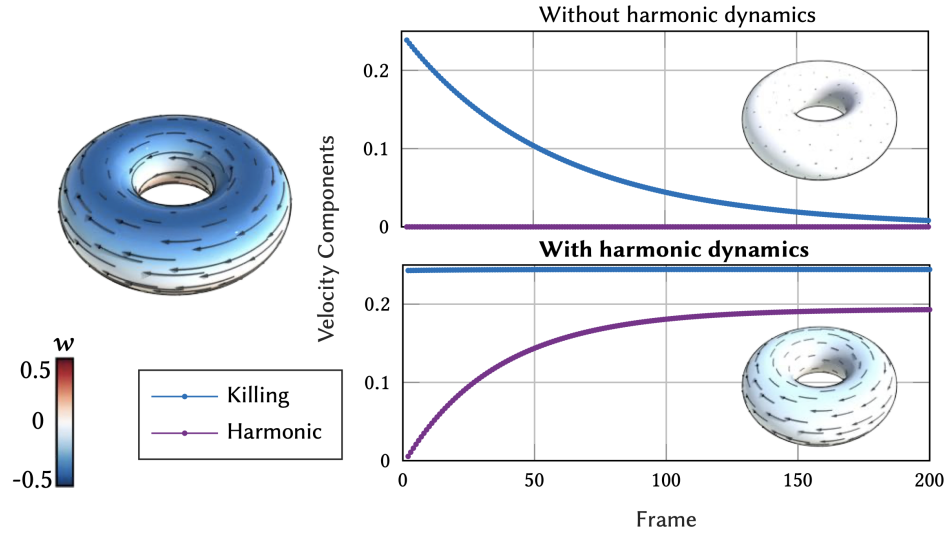


Figure 2.10. Comparison of fluid dynamics on a torus using the full system (2.38) (bottom right), which includes harmonic component evolution, versus a system with harmonic dynamics disabled (top right), both initialized from the same initial condition (left). The plots show the evolution of the Killing field and harmonic field components over time. The full system preserves the Killing component and reveals interaction between the harmonic and vorticity components. In contrast, disabling the harmonic dynamics causes the Killing component to be incorrectly damped out. Here $\nu = 0.1, \Delta t = 0.1$.

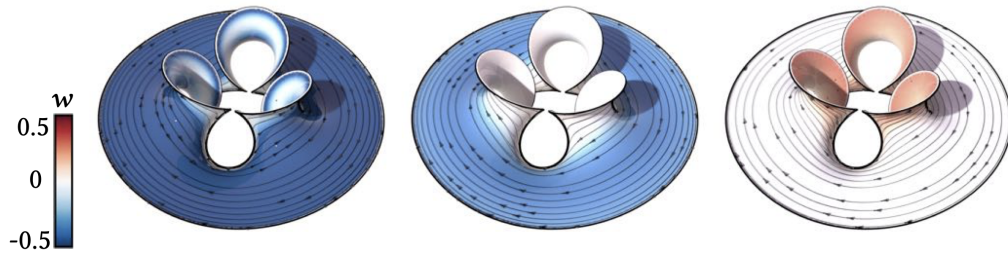


Figure 2.11. Viscous flow simulated on an Enneper surface, which possesses an intrinsic Killing field. The free-slip condition is imposed at the boundary. An initial flow (left) relaxes to the pure Killing field (right). The viscosity is $\nu = 0.1$. The sequence shows frames 2, 50, 300 with $\Delta t = 0.1$.

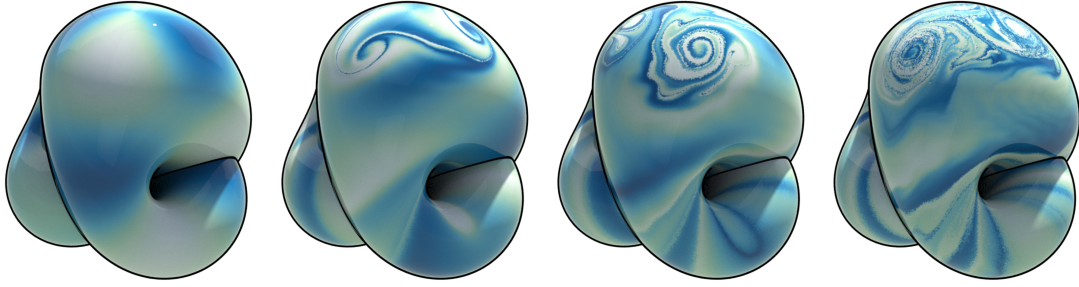


Figure 2.12. Simulation of the Navier–Stokes fluid on the non-orientable *Boy surface* using our vorticity method.

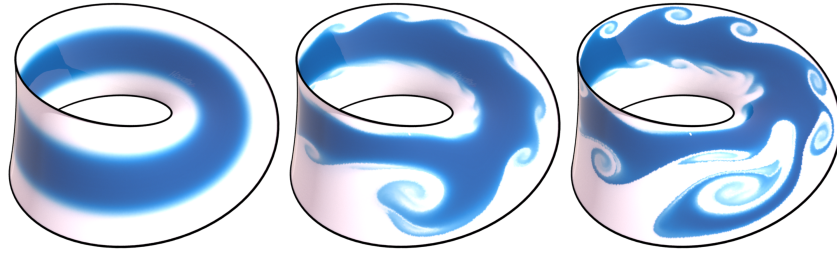


Figure 2.13. Kelvin–Helmholtz instability on a Möbius strip. The initial condition includes a vortex sheet along the boundary of the blue region and a nonzero harmonic component, forming a jet within the blue region. The Möbius strip has width 1, the initial jet speed is 0.1, the viscosity is $\nu = 10^{-5}$, and the time step is $\Delta t = 0.1$. The sequence shows frames 1, 600, 780.

double cover is a topological sphere. The *Möbius strip* in Figure 2.13 is double-covered by a cylinder. The *Klein bottle* in Figure 2.14 is double-covered by a torus. The simulation runs on the oriented double covering mesh with an initial condition that obeys the even/odd parity in Table 2.1.

2.6.4 Boundary conditions

While the previous examples in Figures 2.4, 2.11, and 2.13 demonstrate the behavior of fluids under the free-slip boundary condition, we now examine the influence of friction applied at the boundary.

Figure 2.5 shows that the no-slip boundary condition on a moving boundary—enforced via Algorithm 2—can set motion in an initially stationary viscous fluid.

In Figure 2.15, a small disk is removed from a torus to represent a fluid obstacle.



Figure 2.14. Fluid simulation on a Klein bottle. The initial condition is set with a jet along the tube similar to that of Figure 2.13. The sequence shows frames 1, 50, 300.

The initial condition is set with zero vorticity but a nonzero harmonic component, producing a background flow past the obstacle. When the obstacle boundary is set as free-slip (top row), a small amount of vorticity is generated, but insufficient to form a von Kármán vortex street. In contrast, when the boundary is given high friction—approximated by a near-infinite relative friction coefficient $\frac{\alpha}{4\nu} \gg 1$ to emulating a no-slip boundary (bottom row)—vortex shedding is significantly enhanced, resulting in the formation of a von Kármán vortex street.

Figure 2.16 compares two no-slip boundary treatments: one using a high-friction boundary, as in Figure 2.15 (bottom), and the other enforcing the hard no-slip constraint computed with Algorithm 2. While both approaches yield similar results, the high-friction method is significantly more efficient. Moreover, the fixed-point iteration in Algorithm 2 requires a finer mesh near the boundary to ensure convergence, which is the domain employed for Figure 2.16.

Figure 2.17 compares flow past a high-friction obstacle at two different viscosities, i.e., two different Reynolds numbers. From both Figure 2.15 and Figure 2.17, one can observe that vorticity exhibits greater numerical dispersion (spatial oscillation) at higher Reynolds numbers and lower resolutions. Such dispersion is expected from the discrete operator (2.58) [8]. Note, however, that these dispersive oscillations appear in the vorticity field (a derivative quantity) and have little effect on the velocity field (an

integrated quantity) or on the resulting fluid animation.

Figure 2.18 presents a similar experiment conducted on the Klein bottle.

2.7 Conclusion

We have presented a vorticity formulation of the Navier–Stokes equations on general surfaces, along with a corresponding numerical algorithm. A central contribution is our emphasis on curvature-related terms in the viscous force, which influence both the vorticity equation and the evolution of harmonic components. By comparing our full model with commonly used simplified variants that omit curvature effects or harmonic dynamics, we demonstrate that only the complete system reproduces physically meaningful behavior, such as the conservation of frictionless rigid motions.

Our numerical scheme is conditionally stable, similar to the approaches in [8] and [127], with the advection step subject to a CFL condition. The implicit treatment of the viscous force (Algorithm 1) and the no-slip condition (Algorithm 2) is carried out using fixed-point iterations. While fixed-point methods are simple to implement and converge quickly for the main algorithm (Algorithm 1), they are less efficient for enforcing the no-slip condition. We believe that Newton’s method may be better suited for these implicit steps, and we leave its development for future work.

A notable analytical result of our study is an explicit formula for the jump condition of vorticity across a sheet of singular Gaussian curvature. This mathematical result is used to simplify the enforcement of boundary conditions: by interpreting boundary geodesic curvature, augmented by the friction coefficient, as a singular sheet of Gaussian curvature, the correct vorticity at the boundary naturally emerges through the jump condition. This approach is useful even in flat domains with curved boundaries.

Under free-slip boundary condition, our formulation of the harmonic evolution

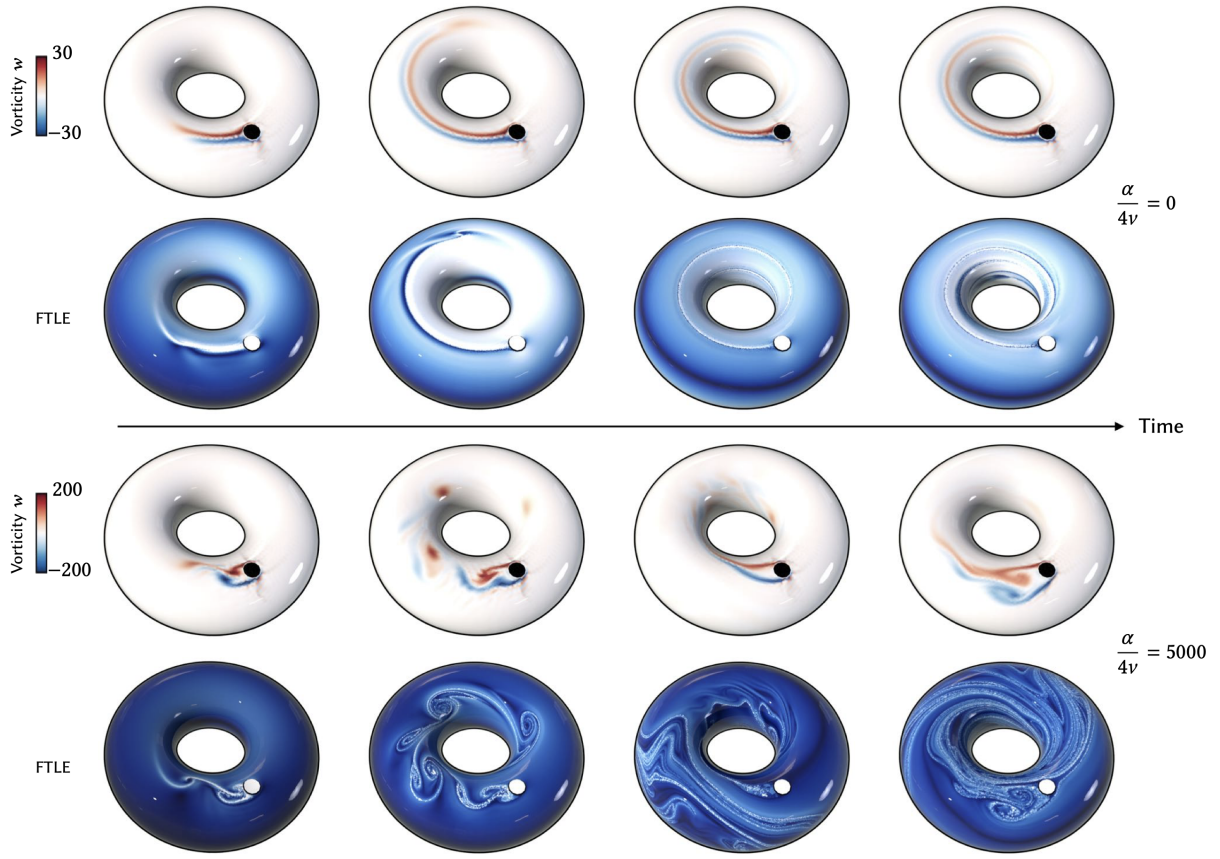


Figure 2.15. Flow past an obstacle on a torus. With identical initial conditions and fluid viscosity, the relative friction coefficient at the boundary affects the extent of vortex shedding. Compared to the free-slip boundary (top row), the near-no-slip friction boundary (bottom row) produces pronounced vortex shedding, leading to the formation of a von Kármán vortex street. The flows are visualized using vorticity and the backward-time Finite-Time Lyapunov Exponent (FTLE) [49]. The torus has radii 1 and 0.5; the initial harmonic component has $c = 10$; the viscosity is $\nu = 10^{-3}$; and the simulation uses $\Delta t = 0.002$. The sequence shows frames 61, 201, 751, 996.

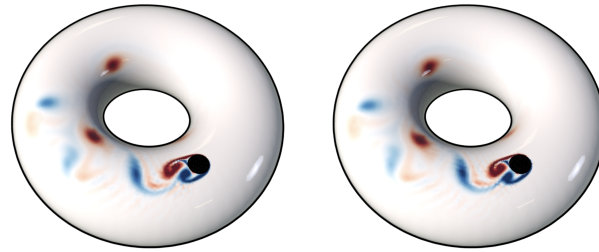


Figure 2.16. Comparison between a high-friction boundary condition ($\frac{\alpha}{4\nu} = 1000$) (left) and the no-slip condition computed with Algorithm 2 (right). Producing similar results, the high-friction boundary is computationally more efficient. The mesh is refined near the boundary. Frame 201 is shown.

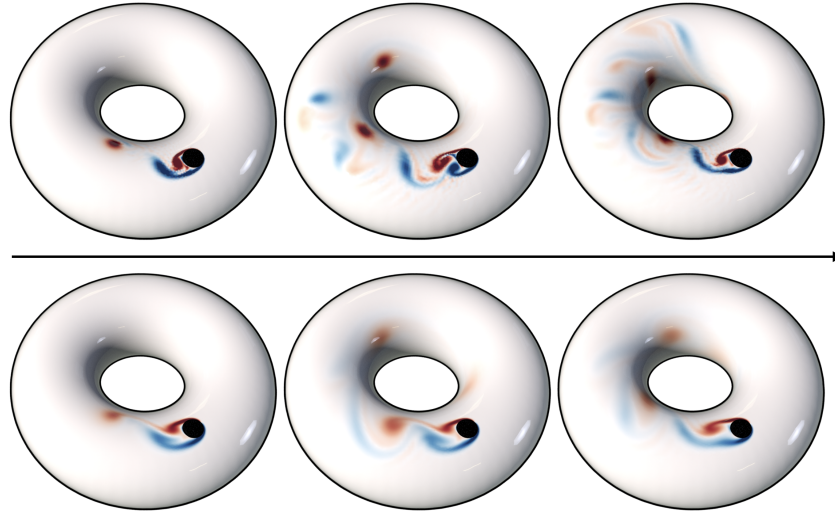


Figure 2.17. Flow past a high-friction ($\frac{\alpha}{4\nu} = 1000$) obstacle on a torus with low viscosity ($\nu = 10^{-3}$, top) and high viscosity ($\nu = 10^{-2}$, bottom). Vortex shedding occurs in both cases, forming von Kármán vortex streets at two different Reynolds numbers. The sequence shows frames 61, 201, 301.

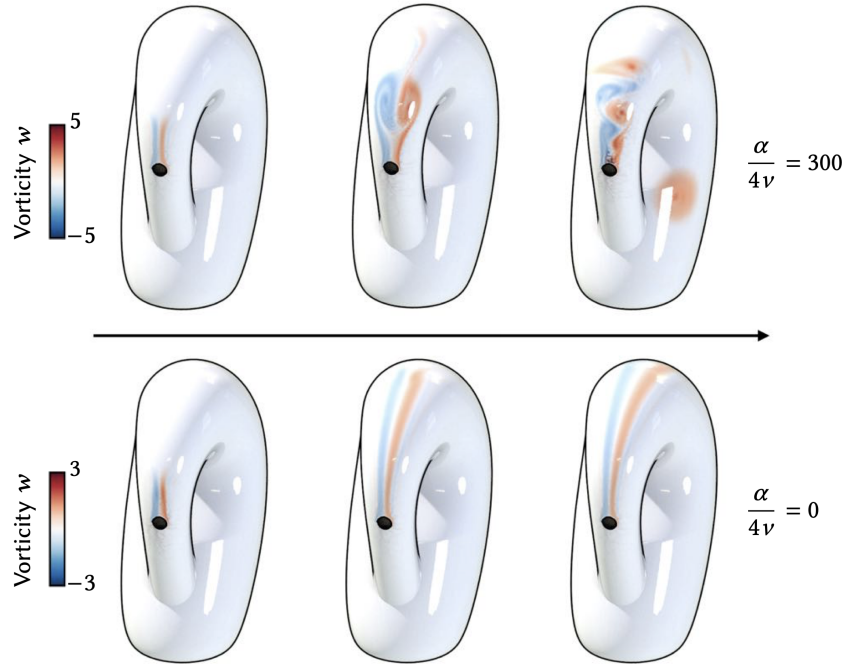


Figure 2.18. Flow past an obstacle on a Klein bottle with identical initial conditions and viscosity. Top row: friction boundary condition produces a von Kármán vortex street. Bottom row: free-slip boundary condition. The initial harmonic component has $c = 0.5$, and the viscosity is $\nu = 10^{-5}$. The sequence shows frames 30, 160, 390.

equation reveals a direct relationship between boundary curvature, boundary velocity, and the dynamics of harmonic components (Corollary 2.3.1). This observation may provide new mathematical insight into the Kutta condition and complements recent work aimed at developing its theoretical foundation [110].

Finally, we describe a general technique for simulating fluids on non-orientable surfaces by constructing their oriented double cover and carefully distinguishing between even and odd types of vector fields and differential forms. This approach, in particular, leads to a natural extension of the computational Hodge decomposition to non-orientable surfaces (Section 2.4.1).

We hope that these contributions will inspire further developments at the intersection of geometry and fluid dynamics.

Appendix

A Killing operator

This appendix elaborates on the Killing operator (Definition 2.2.1) and provides proofs for Propositions 2.2.1, 2.2.2, 2.3.1, and Theorems 2.2.1, 2.2.2.

The proofs of Proposition 2.2.1 and Theorem 2.2.2 rely only on elementary differential geometry and are presented in Section A.1 and Section A.2, respectively.

In contrast, the proofs of Propositions 2.2.2, 2.3.1, and Theorem 2.2.1 become significantly simpler once we establish the relationship between the Killing operator and the *Dolbeault operator* $\bar{\partial}$, under the identification of the tangent bundle TM as a *complex line bundle*, as explained in Section A.3.

A.1 Proof of Proposition 2.2.1

We show that $(\mathcal{K}u)[[a, b]] = \frac{1}{2}(\langle \nabla_a u, b \rangle + \langle \nabla_b u, a \rangle)$, where $\mathcal{K}u := \frac{1}{2} \mathcal{L}_u g$.

Apply the product rule for Lie derivative:

$$\mathcal{L}_u(g[[a, b]]) = (\mathcal{L}_u g)[[a, b]] + g[[\mathcal{L}_u a, b]] + g[[a, \mathcal{L}_u b]]. \quad (2.71)$$

Express the metric as $g[[\cdot, \cdot]] = \langle \cdot, \cdot \rangle$, and use the identity $\mathcal{L}_u a = [u, a] = \nabla_u a - \nabla_a u$, which holds because the Levi-Civita connection ∇ is torsion-free. The left-hand side of (2.71) becomes

$$\mathcal{L}_u \langle a, b \rangle = d_u \langle a, b \rangle = \langle \nabla_u a, b \rangle + \langle a, \nabla_u b \rangle, \quad (2.72)$$

by metric compatibility of ∇ . Substituting this into (2.71), we get:

$$\begin{aligned} \langle \cancel{\nabla_u a}, b \rangle + \langle a, \cancel{\nabla_u b} \rangle &= 2(\mathcal{K}u)[[a, b]] + \langle \cancel{\nabla_u a} - \nabla_a u, b \rangle \\ &\quad + \langle a, \cancel{\nabla_u b} - \nabla_b u \rangle, \end{aligned} \quad (2.73)$$

which implies the desired identity (2.6). $\square$

A.2 Proof of Theorem 2.2.2

We show that the component of the velocity field along a Killing vector field is conserved under the Navier–Stokes equation.

Let $v \in \ker(\mathcal{K})$ be a Killing vector field. By Proposition 2.2.1, it satisfies the identity

$$\langle \nabla_a v, b \rangle = -\langle \nabla_b v, a \rangle. \quad (2.74)$$

Suppose the velocity field u satisfies the Navier–Stokes equation (2.20): $\dot{u} + \nabla_u u = -\text{grad } p + \nu \Delta_V u$. We take the L^2 inner product of both sides of the equation with the Killing field v . The first term yields $\langle \dot{u}, v \rangle = \frac{d}{dt} \langle u, v \rangle$, which is the time

derivative of the projection of $\mathbf{u}$ onto the Killing field $\mathbf{v}$. Our goal is to show that the remaining terms $\langle\langle -\nabla_{\mathbf{u}}\mathbf{u} - \text{grad } p + \nu\Delta_V\mathbf{u}, \mathbf{v} \rangle\rangle$ vanish, thus establishing $\frac{d}{dt}\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle = 0$.

We first observe that $\langle\langle \Delta_V\mathbf{u}, \mathbf{v} \rangle\rangle = 0$. This follows from the fact that Δ_V is the functional gradient of $\mathcal{E}_V[\mathbf{u}] = \|\mathcal{K}\mathbf{u}\|^2$ (Definition 2.2.6), and hence⁸ $\langle\langle \Delta_V\mathbf{u}, \mathbf{v} \rangle\rangle = 2\langle\langle \mathcal{K}\mathbf{u}, \mathcal{K}\mathbf{v} \rangle\rangle = 0$ (since $\mathcal{K}\mathbf{v} = 0$).

Next, we have $\langle\langle -\text{grad } p, \mathbf{v} \rangle\rangle = \langle\langle p, \text{div } \mathbf{v} \rangle\rangle = 0$, since every Killing field is divergence-free.

It remains to show that $\langle\langle \nabla_{\mathbf{u}}\mathbf{u}, \mathbf{v} \rangle\rangle = 0$. Using integration by parts and the divergence-freeness of $\mathbf{u}$, we have $\langle\langle \nabla_{\mathbf{u}}\mathbf{u}, \mathbf{v} \rangle\rangle = -\langle\langle \mathbf{u}, \nabla_{\mathbf{u}}\mathbf{v} \rangle\rangle$, which vanishes by the skew-symmetry of $\nabla\mathbf{v}$ in (2.74). $\square$

A.3 Killing operator in terms of complex analysis on complex manifolds

For an oriented compact Riemannian 2-manifold M , the tangent bundle TM , equipped with the counterclockwise 90° rotation operator J , becomes a *complex line bundle* over M . This complex structure endows M with the structure of a *Riemann surface*. Compact Riemann surfaces are, in fact, *complex algebraic curves*, and as such, they naturally admit several important operators from complex analysis, which we now introduce.

Complex-linear and anti-linear forms

The complex structure J on TM induces a corresponding complex structure on 1-forms, represented by the Hodge star operator $\star_1$ acting on 1-forms. This operator is related to J by the adjoint relation:

$$\langle\alpha|J\mathbf{a}\rangle = \langle-\star_1\alpha|\mathbf{a}\rangle, \quad \alpha \in \Omega^1(M), \mathbf{a} \in \Gamma(TM). \quad (2.75)$$

⁸If $\partial M \neq \emptyset$, the natural boundary condition (i.e., the free-slip condition from Proposition 2.3.1) must be imposed on $\mathbf{u}$.

Equivalently, this means $-\star_1 \alpha = \alpha \circ J$.

For brevity, we will omit the subscript k in the Hodge star $\star_k$ acting on k -forms, unless needed for clarity.

With both TM and T^*M now endowed with complex structures, we can classify vector-valued 1-forms according to whether they preserve or reverse the complex structure.

Definition A.1. A vector-valued 1-form $A \in \Omega^1(M, TM)$ is said to be:

- **complex-linear** if $-\star A = A \circ J = J \circ A$.
- **complex-antilinear** if $-\star A = A \circ J = -J \circ A$.

The corresponding spaces of complex-(anti)linear vector-valued 1-forms are:

$$\Gamma(K \otimes TM) := \{A \in \Omega^1(M, TM) \mid -\star A = JA\} \quad (2.76)$$

$$\Gamma(\bar{K} \otimes TM) := \{A \in \Omega^1(M, TM) \mid \star A = JA\} \quad (2.77)$$

Here, K denotes the *canonical bundle* of the algebraic curve M .

Every vector-valued 1-form admits a decomposition into complex-linear and a complex-antilinear components:

$$\Omega^1(M, TM) = \Gamma(K \otimes TM) \oplus \Gamma(\bar{K} \otimes TM), \quad (2.78a)$$

$$A = \underbrace{\frac{1}{2}(A + J\star A)}_{\in \Gamma(K \otimes TM)} + \underbrace{\frac{1}{2}(A - J\star A)}_{\in \Gamma(\bar{K} \otimes TM)}. \quad (2.78b)$$

These two componets are orthogonal under the Frobenius product:

$$\langle A \wedge \star B \rangle = 0 \quad \forall A \in \Gamma(K \otimes TM), B \in \Gamma(\bar{K} \otimes TM). \quad (2.79)$$

The complex-linear and complex-antilinear tensors can also be understood in

terms of symmetric and skew-symmetric bilinear forms. Applying the $\flat$ operator to a vector-valued 1-form yields a covector-valued 1-form, (i.e., a bilinear form), defined by

$$A^\flat \llbracket \mathbf{a}, \mathbf{b} \rrbracket := \langle A\mathbf{a}, \mathbf{b} \rangle. \quad (2.80)$$

One can verify that

$$A \in \Gamma(\bar{K} \otimes TM) \text{ if and only if } A^\flat \text{ is symmetric and } \text{tr}(A) = 0. \quad (2.81)$$

In fact, the complex-linear part $A' := \frac{1}{2}(A + J \star A)$ and the complex-antilinear part $A'' := \frac{1}{2}(A - J \star A)$ correspond respectively to the trace/skew-symmetric part and the trace-free symmetric part of $A^\flat$:

$$(A')^\flat \llbracket \mathbf{a}, \mathbf{b} \rrbracket = \frac{1}{2} \text{tr}(A) \langle \mathbf{a}, \mathbf{b} \rangle + \frac{1}{2} (\langle A\mathbf{a}, \mathbf{b} \rangle - \langle A\mathbf{b}, \mathbf{a} \rangle), \quad (2.82a)$$

$$(A'')^\flat \llbracket \mathbf{a}, \mathbf{b} \rrbracket = \frac{1}{2} (\langle A\mathbf{a}, \mathbf{b} \rangle + \langle A\mathbf{b}, \mathbf{a} \rangle) - \frac{1}{2} \text{tr}(A) \langle \mathbf{a}, \mathbf{b} \rangle. \quad (2.82b)$$

These expressions can be verified by evaluating on vectors from a (pointwise) orthonormal basis.

Decomposition of vector gradients

Vector-valued 1-forms are fundamental objects in the analysis of flow fields, as they naturally arise when differentiating a vector field such as the velocity. Specifically, the Levi-Civita covariant derivative of a vector field $\mathbf{u} \in \Gamma(TM)$ is a vector-valued 1-form $\nabla \mathbf{u} \in \Omega^1(M, TM)$.

Applying the decomposition from (2.78) to $\nabla \mathbf{u}$, we obtain its complex-linear and complex-antilinear components, denoted by $\partial \mathbf{u}$ and $\bar{\partial} \mathbf{u}$ respectively.

Definition A.2. *The Wirtinger differential operators $\partial: \Gamma(TM) \rightarrow \Gamma(K \otimes TM)$ and*

$\bar{\partial}: \Gamma(TM) \rightarrow \Gamma(\bar{K} \otimes TM)$ (the latter also known as the **Dolbeault operator**) are defined as

$$\partial \mathbf{u} := \frac{1}{2}(\nabla \mathbf{u} + J \star \nabla \mathbf{u}) \quad (2.83)$$

$$\bar{\partial} \mathbf{u} := \frac{1}{2}(\nabla \mathbf{u} - J \star \nabla \mathbf{u}), \quad (2.84)$$

which are the complex-linear and complex-antilinear parts of $\nabla \mathbf{u}$:

$$\nabla \mathbf{u} = \partial \mathbf{u} + \bar{\partial} \mathbf{u}. \quad (2.85)$$

By (2.82), Proposition 2.2.1, the fact that $\text{tr}(\nabla \mathbf{u}) = \text{div}(\mathbf{u})$, and the fact that $d(\mathbf{u}^\flat)$ corresponds to twice the skew-symmetric part⁹ of $\nabla \mathbf{u}$, we obtain:

$$(\partial \mathbf{u})^\flat = \frac{1}{2}(\text{div} \mathbf{u})g + \frac{1}{2}d(\mathbf{u}^\flat) \quad (2.86)$$

$$(\bar{\partial} \mathbf{u})^\flat = \mathcal{K} \mathbf{u} - \frac{1}{2}(\text{div} \mathbf{u})g. \quad (2.87)$$

If $\mathbf{u}$ is a divergence-free velocity field, i.e., $\mathbf{u} \in \mathfrak{X}_{\text{div}}$, then the ∂ and $\bar{\partial}$ derivatives of $\mathbf{u}$ reduce to:

$$\boxed{(\partial \mathbf{u})^\flat = \frac{1}{2}\omega} \quad \text{and} \quad \boxed{(\bar{\partial} \mathbf{u})^\flat = \mathcal{K} \mathbf{u}} \quad (2.88)$$

where $\omega = d(\mathbf{u}^\flat)$ is the vorticity 2-form, and $\mathcal{K} \mathbf{u}$ is the strain rate tensor of the flow.

The identities in (2.88) allow us to use the tools of ∂ - and $\bar{\partial}$ -calculus from complex differential geometry to perform calculations involving vorticity and the strain rate.

⁹To verify that $d(\mathbf{u})^\flat[[\mathbf{a}, \mathbf{b}]] = \langle \nabla_{\mathbf{a}} \mathbf{u}, \mathbf{b} \rangle - \langle \nabla_{\mathbf{b}} \mathbf{u}, \mathbf{a} \rangle$, expand the expression $d(\mathbf{u})^\flat[[\mathbf{a}, \mathbf{b}]] = d_{\mathbf{a}} \langle \mathbf{u}, \mathbf{b} \rangle - d_{\mathbf{b}} \langle \mathbf{u}, \mathbf{a} \rangle - \langle \mathbf{u}, [\mathbf{a}, \mathbf{b}] \rangle$, and apply the metric compatibility property $d_{\mathbf{a}} \langle \mathbf{u}, \mathbf{b} \rangle = \langle \nabla_{\mathbf{a}} \mathbf{u}, \mathbf{b} \rangle + \langle \mathbf{u}, \nabla_{\mathbf{a}} \mathbf{b} \rangle$ (similarly for the second term) along with the torsion-free property $[\mathbf{a}, \mathbf{b}] = \nabla_{\mathbf{a}} \mathbf{b} - \nabla_{\mathbf{b}} \mathbf{a}$.

A.4 Proof of Proposition 2.2.2

We show that the functional gradient of the Rayleigh dissipation functional $\mathcal{R}: \mathfrak{X}_{\text{div}} \rightarrow \mathbb{R}$, defined by $\mathcal{R}[\mathbf{u}] = \iint_M \nu |\mathcal{K}\mathbf{u}|^2 dA$, is given by $(\text{grad } \mathcal{R})|_{\mathbf{u}} = -2\nu \text{div}(\mathcal{K}\mathbf{u})^\sharp$. Using the identity from (2.88), this is equivalent to showing $(\text{grad } \mathcal{R})|_{\mathbf{u}} = -2\nu \star_0^{-1} d^\nabla \star_1 (\bar{\partial}\mathbf{u})$, where d^∇ denotes the covariant exterior derivative acting on vector-valued forms.

By (2.88), the functional $\mathcal{R}[\mathbf{u}]$ can be written in terms of the Dolbeault operator as

$$\mathcal{R}[\mathbf{u}] = \nu \iint_M \langle \bar{\partial}\mathbf{u} \wedge \star \bar{\partial}\mathbf{u} \rangle. \quad (2.89)$$

Let $\dot{\mathbf{u}} \in \mathfrak{X}_{\text{div}}$ be a divergence-free variation vanishing on ∂M . The variation of $\mathcal{R}$ in the direction $\dot{\mathbf{u}}$ is

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{R}[\mathbf{u} + \epsilon \dot{\mathbf{u}}] = 2\nu \iint_M \langle \bar{\partial}\dot{\mathbf{u}} \wedge \star \bar{\partial}\mathbf{u} \rangle \quad (2.90)$$

$$= 2\nu \iint_M \langle \nabla \dot{\mathbf{u}} \wedge \star \bar{\partial}\mathbf{u} \rangle, \quad (2.91)$$

where the second equality uses the decomposition (2.85) and the orthogonality of complex-linear and antilinear parts from (2.79). We now integrate by parts:

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{R}[\mathbf{u} + \epsilon \dot{\mathbf{u}}] = -2\nu \iint_M \langle \dot{\mathbf{u}}, d^\nabla \star \bar{\partial}\mathbf{u} \rangle \quad (2.92)$$

$$= -2\nu \iint_M \langle \dot{\mathbf{u}}, \star_0^{-1} d^\nabla \star_1 \bar{\partial}\mathbf{u} \rangle dA \quad (2.93)$$

$$= \langle \dot{\mathbf{u}}, -2\nu \star_0^{-1} d^\nabla \star_1 \bar{\partial}\mathbf{u} \rangle. \quad (2.94)$$

Therefore, we conclude that $(\text{grad } \mathcal{R})|_{\mathbf{u}} = -2\nu \star_0^{-1} d^\nabla \star_1 \bar{\partial}\mathbf{u}$. $\square$

A.5 Proof of Proposition 2.3.1

We derive the natural boundary condition associated with the functional (2.8), which can also be expressed in the form (2.89). To do so, we repeat the derivation in Section A.4, but this time without assuming that the variation $\dot{\mathbf{u}}$ vanishes on the boundary ∂M , thereby retaining and examining the boundary terms.

Specifically, the integration by parts in (2.91) now yields:

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \mathcal{R}[\mathbf{u} + \epsilon \dot{\mathbf{u}}] = 2\nu \oint_{\partial M} \langle \dot{\mathbf{u}}, \star \bar{\partial} \mathbf{u} \rangle - 2\nu \iint_M \langle \dot{\mathbf{u}}, d^\nabla \star \bar{\partial} \mathbf{u} \rangle. \quad (2.95)$$

Therefore, in order for the identity $(\text{grad } \mathcal{R})|_{\mathbf{u}} = -2\nu \text{div}(\mathcal{K}\mathbf{u})^\sharp$ to hold for all divergence-free variations $\dot{\mathbf{u}} \in \mathfrak{X}_{\text{div}}$ that may be nonzero on ∂M , the boundary term must vanish:

$$\left\langle \mathbf{t}, (\star \bar{\partial} \mathbf{u}) \llbracket \mathbf{t} \rrbracket \right\rangle = 0 \quad \text{on } \partial M, \quad (2.96)$$

where $\mathbf{t}$ denotes the unit tangent vector along the boundary curve ∂M . Using the identity (2.75) and setting $\mathbf{n} = J\mathbf{t}$ as the inward-pointing unit normal, condition (2.96) becomes:

$$-\left\langle \mathbf{t}, (\bar{\partial} \mathbf{u}) \llbracket \mathbf{n} \rrbracket \right\rangle = 0 \quad \text{on } \partial M. \quad (2.97)$$

By the identity (2.88), this is equivalent to $(\mathcal{K}\mathbf{u}) \llbracket \mathbf{t}, \mathbf{n} \rrbracket = 0$. □

A.6 Proof of Theorem 2.2.1

Using (2.88), we can rewrite the Dirichlet energies from Section 2.2.2 for divergence-free vector fields $\mathbf{u} \in \mathfrak{X}_{\text{div}}$ as:

$$\mathcal{E}_D[\mathbf{u}] = \frac{1}{2} \iint_M \langle \nabla \mathbf{u} \wedge \star \nabla \mathbf{u} \rangle, \quad (2.98)$$

$$\mathcal{E}_H[\mathbf{u}] = \iint_M \langle \partial \mathbf{u} \wedge \star \partial \mathbf{u} \rangle, \quad (2.99)$$

$$\mathcal{E}_V[\mathbf{u}] = \iint_M \langle \bar{\partial} \mathbf{u} \wedge \star \bar{\partial} \mathbf{u} \rangle. \quad (2.100)$$

Following a similar derivation as in Section A.4, we find that the (negative) functional gradients of these Dirichlet energies—namely, the Bochner Laplacian (Definition 2.2.7), the Hodge Laplacian (Definition 2.2.8), and the viscous Laplacian (Definition 2.2.9)—can be expressed as:

$$\star_0 \Delta_B \mathbf{u} = d^\nabla \star_1 \nabla \mathbf{u}, \quad (2.101)$$

$$\star_0 \Delta_H \mathbf{u} = 2d^\nabla \star_1 \partial \mathbf{u} = -2Jd^\nabla \partial \mathbf{u}, \quad (2.102)$$

$$\star_0 \Delta_V \mathbf{u} = 2d^\nabla \star_1 \bar{\partial} \mathbf{u} = 2Jd^\nabla \bar{\partial} \mathbf{u}. \quad (2.103)$$

The second equalities in (2.102) and (2.103) follow from the complex-linear and antilinear types of $\partial \mathbf{u}$ and $\bar{\partial} \mathbf{u}$ (see Definition A.1 and Definition A.2), and from the identity $d^\nabla \circ J = J \circ d^\nabla$, which holds due to the metric compatibility of the Levi-Civita connection ∇ .

Substituting (2.83) into (2.102) yields:

$$-2Jd^\nabla \partial \mathbf{u} = -Jd^\nabla (\nabla \mathbf{u} + J \star \nabla \mathbf{u}) \quad (2.104)$$

$$= -Jd^\nabla \nabla \mathbf{u} + d^\nabla \star \nabla \mathbf{u} \quad (2.105)$$

$$= -\star_0 K \mathbf{u} + d^\nabla \star_1 \nabla \mathbf{u}, \quad (2.106)$$

where we used $d^\nabla \nabla u = -J\Omega^\nabla u$, with $\Omega^\nabla \in \Omega^2(M)$ denoting the curvature 2-form of the connection ∇ . For the Levi-Civita connection, this curvature form is the Gaussian curvature density $\star_0 K$.

Similarly, substituting (2.84) into (2.103) gives:

$$2Jd^\nabla \bar{\partial} u = \star_0 K u + d^\nabla \star_1 \nabla u. \quad (2.107)$$

Thus, we obtain the relations:

$$\Delta_H u = \Delta_B u - K u, \quad \Delta_V u = \Delta_B u + K u. \quad (2.108)$$

□

B Derivation for the jump conditions

This appendix provides the background necessary to establish Theorem 2.2.3, which states that the vorticity function exhibits a jump across curvature sheets.

Let $\Gamma \in M$ be an oriented curve. The *positive* (resp. *negative*) side of Γ is defined as the left (resp. right) side when traversing Γ in its oriented direction.

If a function u has a discontinuity along Γ , we denote the jump by $[u]_\Gamma: \Gamma \rightarrow \mathbb{R}$, defined as the value of u on the positive side minus its value on the negative side.

The normal vector n to Γ is defined by $n = Jt$, where t is the unit tangent to Γ ; this convention agrees with the inward unit normal when $\Gamma = \partial M$.

Let u be a function that is smooth on $M \setminus \Gamma$ but potentially discontinuous along Γ , and let φ be an arbitrary smooth test function on M . By applying Green's identity,

we obtain:

$$\begin{aligned} \iint_{M \setminus \Gamma} (\Delta u) \varphi \, dA &= \int_{\Gamma} \left([u]_{\Gamma} \frac{\partial \varphi}{\partial n} - \left[\frac{\partial u}{\partial n} \right]_{\Gamma} \varphi \right) ds \\ &+ \oint_{\partial M} \left(u \frac{\partial \varphi}{\partial n} - \frac{\partial u}{\partial n} \varphi \right) ds + \iint_M u \Delta \varphi \, dA. \end{aligned} \quad (2.109)$$

Eq. (2.109) implies that the (weak) Laplacian of u over the entire domain M , defined via

$$\iint_M (\Delta u) \varphi \, dA \equiv \oint_{\partial M} \left(f \frac{\partial \varphi}{\partial n} - \frac{\partial u}{\partial n} \varphi \right) ds + \iint_M u \Delta \varphi \, dA, \quad (2.110)$$

can be evaluated as

$$\boxed{\iint_M (\Delta u) \varphi \, dA = \iint_{M \setminus \Gamma} (\Delta u) \varphi \, dA + \int_{\Gamma} \left(\left[\frac{\partial u}{\partial n} \right]_{\Gamma} \varphi - [u]_{\Gamma} \frac{\partial \varphi}{\partial n} \right) ds.} \quad (2.111)$$

In other words, the Laplacian Δu can be decomposed as $\Delta u = (\Delta u)_{\text{reg}} + (\Delta u)_{\text{sing}}$, where $(\Delta u)_{\text{reg}}$ is the classical Laplacian of u away from the curve, and $(\Delta u)_{\text{sing}}$ is a distribution supported on Γ , defined by the property $\iint (\Delta u)_{\text{sing}} \varphi = \int_{\Gamma} \left(\left[\frac{\partial u}{\partial n} \right]_{\Gamma} \varphi - [u]_{\Gamma} \frac{\partial \varphi}{\partial n} \right) ds$.

We now express this distributional singularity using Dirac- δ distributions:

Definition B.1. Let $h: \Gamma \rightarrow \mathbb{R}$ be a function defined on Γ . The h -weighted δ function on Γ is the distribution $h\delta_{\Gamma}$ defined by

$$\iint_M (h\delta_{\Gamma}) \varphi \, dA := \int_{\Gamma} h \varphi \, ds, \quad \forall \varphi \in C^{\infty}(M). \quad (2.112)$$

The h -weighted δ' function on Γ is the distribution $h\delta'_{\Gamma}$ defined by

$$\iint_M (h\delta'_{\Gamma}) \varphi \, dA := \int_{\Gamma} -h \frac{\partial \varphi}{\partial n} \, ds, \quad \forall \varphi \in C^{\infty}(M). \quad (2.113)$$

Using the notation from Definition B.1 and eq. (2.109), we arrive at the following

general expression for the Laplacian of a function with a jump across Γ :

$$\Delta u = (\Delta u)_{\text{reg}} + \left[\frac{\partial u}{\partial \mathbf{n}} \right]_{\Gamma} \delta_{\Gamma} + [u]_{\Gamma} \delta'_{\Gamma}. \quad (2.114)$$

B.1 Proof of Theorem 2.2.3

We show that the vorticity equation (2.28) $\frac{Dw}{Dt} = \nu \Delta w + 2\nu \text{curl}(K\mathbf{u})$ with singular Gaussian curvature $K = K_{\text{reg}} + f\delta_{\Gamma}$ gives rise to a jump condition $[w]_{\Gamma} = 2f\langle \mathbf{u}, \mathbf{t} \rangle$ along the curve Γ .

Note that the term $2\nu \text{curl}(K\mathbf{u})$ is a distribution. For any test function $\varphi \in C^{\infty}(M)$, we have

$$\iint_M 2\nu \text{curl}(K\mathbf{u}) \varphi \, dA = \iint_M -2\nu \text{div}(KJ\mathbf{u}) \varphi \, dA \quad (2.115)$$

$$= 2\nu \iint_M K(J\mathbf{u}) \cdot \text{grad} \varphi \, dA \quad (2.116)$$

$$= 2\nu \iint_M (K_{\text{reg}} + f\delta_{\Gamma})(J\mathbf{u}) \cdot \text{grad} \varphi \, dA \quad (2.117)$$

$$= 2\nu \iint_{M \setminus \Gamma} K_{\text{reg}}(J\mathbf{u}) \cdot \text{grad} \varphi \, dA + 2\nu \oint_{\Gamma} f \langle \mathbf{u}, -J \text{grad} \varphi \rangle \, ds \quad (2.118)$$

$$= 2\nu \iint_M \text{curl}(K_{\text{reg}}\mathbf{u}) \varphi \, dA + 2\nu \oint_{\Gamma} f \langle \mathbf{u}, -J \text{grad} \varphi \rangle \, ds. \quad (2.119)$$

We simplify the boundary integral:

$$\oint_{\Gamma} f \langle \mathbf{u}, -J \text{grad} \varphi \rangle \, ds = \oint_{\Gamma} f \langle \mathbf{u}, \mathbf{t} \rangle \frac{\partial \varphi}{\partial \mathbf{n}} \, ds - \oint_{\Gamma} f \langle \mathbf{u}, \mathbf{n} \rangle \frac{\partial \varphi}{\partial \mathbf{t}} \, ds \quad (2.120)$$

$$= \oint_{\Gamma} f \langle \mathbf{u}, \mathbf{t} \rangle \frac{\partial \varphi}{\partial \mathbf{n}} \, ds + \oint_{\Gamma} \frac{\partial}{\partial \mathbf{t}} (f \langle \mathbf{u}, \mathbf{n} \rangle) \varphi \, ds. \quad (2.121)$$

Thus, the singular part of $2\nu \text{curl}(K\mathbf{u})$ is given by

$$(2\nu \text{curl}(K\mathbf{u}))_{\text{sing}} = -2\nu f \langle \mathbf{u}, \mathbf{t} \rangle \delta'_{\Gamma} + 2\nu \frac{\partial}{\partial \mathbf{t}} (f \langle \mathbf{u}, \mathbf{n} \rangle) \delta_{\Gamma}. \quad (2.122)$$

We now collect the regular and singular terms in the vorticity equation $\frac{Dw}{Dt} =$

$\nu \Delta w + 2\nu \operatorname{curl}(Ku)$, using the decompositions from (2.114) and (2.122):

$$\frac{Dw}{Dt} = \nu ((\Delta w)_{\text{reg}} + 2 \operatorname{curl}(K_{\text{reg}}u)) \quad (2.123)$$

$$+ \nu \left(\left[\frac{\partial w}{\partial \mathbf{n}} \right]_{\Gamma} + 2 \frac{\partial}{\partial t} (f \langle \mathbf{u}, \mathbf{n} \rangle) \right) \delta_{\Gamma} \quad (2.124)$$

$$+ \nu ([w]_{\Gamma} - 2f \langle \mathbf{u}, \mathbf{t} \rangle) \delta'_{\Gamma}. \quad (2.125)$$

For this convection-diffusion equation to admit a bounded solution w , the most singular term δ'_{Γ} (the weight of δ'_{Γ}) must vanish.¹⁰ Hence, we conclude $[w]_{\Gamma} = 2f \langle \mathbf{u}, \mathbf{t} \rangle$ along Γ . $\square$

C Derivation for boundary conditions

C.1 Proof of Theorem 2.3.1

We show that under the natural (free-slip) boundary condition $(Ku) \llbracket \mathbf{t}, \mathbf{n} \rrbracket = 0$ on ∂M , the boundary vorticity satisfies $w_{\partial} = 2\kappa_g u_{\partial}$, where $u_{\partial} = \langle \mathbf{u}, \mathbf{t} \rangle$ is the tangential component of the velocity along ∂M , and κ_g is the geodesic curvature of ∂M .

Expanding the boundary condition $(Ku) \llbracket \mathbf{t}, \mathbf{n} \rrbracket = 0$ using the identity (2.6) gives

$$\langle \nabla_{\mathbf{t}} \mathbf{u}, \mathbf{n} \rangle + \langle \nabla_{\mathbf{n}} \mathbf{u}, \mathbf{t} \rangle = 0 \quad \text{on } \partial M. \quad (2.126)$$

We simplify the first term:

$$\langle \nabla_{\mathbf{t}} \mathbf{u}, \mathbf{n} \rangle = d_{\mathbf{t}} \langle \mathbf{u}, \mathbf{n} \rangle - \langle \mathbf{u}, \nabla_{\mathbf{t}} \mathbf{n} \rangle = \langle \mathbf{u}, \mathbf{t} \rangle \kappa_g = u_{\partial} \kappa_g. \quad (2.127)$$

¹⁰This can be justified rigorously by the method of dominant balance. As ϵ -mollified δ -functions tend to singular distributions, the solution behavior for $t > O(\epsilon^{\alpha})$ (for some power α) becomes asymptotically governed by the equation without the most singular part, requiring the weight of δ'_{Γ} to vanish.

Therefore, the boundary condition becomes:

$$\langle \nabla_n \mathbf{u}, \mathbf{t} \rangle + \kappa_g u_\partial = 0. \quad (2.128)$$

We now express this condition in terms of the vorticity $\omega = d\mathbf{u}^\flat = \star w$. To compute $w = \star_0^{-1} \omega$ on the boundary, we evaluate $\omega \llbracket \mathbf{t}, \mathbf{n} \rrbracket$. This evaluation involves the Lie brackets of $\mathbf{t}$ and $\mathbf{n}$, so we extend both vector fields arbitrarily to a neighborhood of ∂M . Then, at the boundary:

$$w = \omega \llbracket \mathbf{t}, \mathbf{n} \rrbracket = (d\mathbf{u}^\flat) \llbracket \mathbf{t}, \mathbf{n} \rrbracket \quad (2.129)$$

$$= d_t \langle \mathbf{u}, \mathbf{n} \rangle - d_n \langle \mathbf{u}, \mathbf{t} \rangle - \langle \mathbf{u}, [\mathbf{t}, \mathbf{n}] \rangle \quad (2.130)$$

$$= -\langle \nabla_n \mathbf{u}, \mathbf{t} \rangle - \langle \mathbf{u}, \nabla_n \mathbf{t} \rangle - \langle \mathbf{u}, \nabla_t \mathbf{n} - \nabla_n \mathbf{t} \rangle \quad (2.131)$$

$$= 2\kappa_g u_\partial \quad (2.132)$$

where we have substituted (2.128) and applied $\langle \mathbf{u}, \nabla_t \mathbf{n} \rangle = -u_\partial \kappa_g$. $\square$

C.2 Proof of Theorem 2.3.2

We show that the friction boundary condition $2\nu(\mathcal{K}\mathbf{u}) \llbracket \mathbf{t}, \mathbf{n} \rrbracket = \alpha u_\partial$ implies the boundary vorticity formula $w_\partial = (2\kappa_g - \frac{\alpha}{2\nu})u_\partial$ following the same derivation as in Section C.1.

Expanding the condition $2\nu(\mathcal{K}\mathbf{u}) \llbracket \mathbf{t}, \mathbf{n} \rrbracket = \alpha u_\partial$ yields $\langle \nabla_t \mathbf{u}, \mathbf{n} \rangle + \langle \nabla_n \mathbf{u}, \mathbf{t} \rangle - \frac{\alpha}{2\nu} u_\partial = 0$. By (2.127), this implies

$$\langle \nabla_n \mathbf{u}, \mathbf{t} \rangle + (\kappa_g - \frac{\alpha}{2\nu})u_\partial = 0. \quad (2.133)$$

Now, following the computation in (2.129), the boundary vorticity becomes:

$$w = -\langle \nabla_n \mathbf{u}, \mathbf{t} \rangle - \underbrace{\langle \mathbf{u}, \nabla_t \mathbf{n} \rangle}_{u_{\partial} \kappa_g} \stackrel{(2.133)}{=} (2\kappa_g - \frac{\alpha}{2\nu}) u_{\partial}. \quad (2.134)$$

□

C.3 Proof of Theorem 2.3.3

Here, we prove our revised form of the *Quartapelle–Valz–Gris condition*, which states that under the no-slip condition $u_{\partial} = 0$, the vorticity w and the harmonic coefficients $(c_1, \dots, c_m)$ must satisfy

$$\iint_M \phi_f w \, dA + \sum_{i=1}^m c_i \oint_{\partial M} f h_{i\partial} \, ds = 0 \quad \forall f: \partial M \rightarrow \mathbb{R}, \quad (2.135)$$

where ϕ_f denotes the harmonic extension of f to the interior of M .

Restricting the velocity field $\mathbf{u} = -J \operatorname{grad} \psi + \sum_{i=1}^m c_i \mathbf{h}_i$ (see (2.31)) to the boundary and imposing the condition $u_{\partial} = 0$, we obtain

$$\frac{\partial \psi}{\partial \mathbf{n}} + \sum_{i=1}^m c_i h_{i\partial} = 0 \quad (2.136)$$

where $\mathbf{n} = J\mathbf{t}$ is the inward-pointing unit normal along ∂M . Additionally, from (2.32), we have $-\Delta \psi = w$ in M , with Dirichlet boundary condition $\psi|_{\partial M} = 0$. Applying Green's identity gives:

$$\iint_M \phi_f w \, dA = \iint_M \phi_f (-\Delta \psi) \, dA \quad (2.137)$$

$$= \iint_M (-\Delta \phi_f) \psi + \oint_{\partial M} (\phi_f \frac{\partial \psi}{\partial \mathbf{n}} - \frac{\partial \phi_f}{\partial \mathbf{n}} \psi) \, ds. \quad (2.138)$$

Using the facts that $\Delta \phi_f = 0$, $\phi_f|_{\partial M} = f$, and $\psi|_{\partial M} = 0$, along with (2.136), this simplifies to $\iint_M \phi_f w \, dA = \oint_{\partial M} f (-\sum_{i=1}^m c_i h_{i\partial}) \, ds$, which implies the identity in (2.135). □

Chapter 3

Active nematic fluids on curved surfaces

This chapter extends the classical fluid models in Chapters 1 and 2 to active nematic fluids on surfaces, incorporating internal orientational order and activity. This chapter is based on the paper *Active Nematic Fluids on Riemannian 2-Manifolds* by Cuncheng Zhu, David Saintillan, and Albert Chern. The dissertation author was the primary investigator and lead author of this work.

3.1 Introduction

Active nematics, a unique class of systems defined by the interaction between intrinsic activity and orientational order, have emerged as a captivating paradigm to explore complex dynamical phenomena across scales. In biophysical contexts, deciphering the dynamics of these active matter systems offers profound insights into a myriad of processes, encompassing collective dynamics in motile bacterial suspensions, structural organization in cytoskeletal assemblies, the dynamics of self-propelled colloidal particles, and the processes governing tissue morphogenesis [3, 34]. Recent experimental discoveries and the potential for significant advances in fields like material science and biophysics have sparked a growing interest in understanding the intricate connections between the dynamic behavior of active nematic systems and the geometric and topo-

logical constraints imposed by spatial confinement [40, 5, 69, 51, 10, 1, 2, 129, 57, 76, 77]. In this paper, we derive a minimal model for active nematic systems on Riemannian surfaces from first principles using a geometric framework. This formulation allows us to develop a discrete theory and a corresponding numerical scheme that are efficient for general geometries. Our focus is on understanding how nematic order parameters relate to the intrinsic curvature and global topological properties of the underlying space. We expect the method to be a useful tool to study active nematic systems where the local curvature and global topology of the system are important, such as in morphogenesis, and to explore potential engineering applications in soft robotics and morphing materials.

To characterize nematic fields, the prevailing approach in the existing literature relies on the Landau-de Gennes Q -tensor theory, which describes the configuration of the nematic field on a surface via a second-order symmetric tensor. However, in 2D, there exists an alternative representation through a homogeneous quadratic complex scalar function [45, 120, 119]. This approach offers distinct advantages because of the inherent constraints on its degrees of freedom. Unlike the complex representation, the evolution of the Q -tensor needs to be confined within the realm of traceless and symmetric matrices, involving additional techniques such as Lagrange multipliers [109, 105] or projection [55, 13]. Furthermore, the complex representation facilitates a natural extension to k -atic fields (i.e., director fields with k -rotational symmetry) by simply raising the exponent of the complex function to k . Recently, there has been significant interest in modeling tissue dynamics through k -atic liquid crystal theory, highlighting the practical relevance of this approach [45, 120].

The mathematical formulation of nematohydrodynamics, or more broadly of complex fluid flows, has been extensively explored in the literature [32, 14, 12, 104, 72]. Classic approaches primarily rely on an explicit approach, employing coordinate-based expressions and standard vector calculus. They often focus on the mechanics of specific

physical objects in flow and root their arguments in the principle of material frame invariance [72]. These approaches become particularly challenging when attempting to translate ordinary vector calculus identities to curved surfaces. However, in the realm of differential geometry, there exists an abstract yet powerful geometric language, exemplified by the Lie derivative, which functions independently of coordinate systems. This language offers a natural way to comprehend various physical phenomena and conservation laws in fluid dynamics [43]. Concepts such as convected time derivatives in complex fluids can be effectively generalized through the use of the Lie derivative, unifying the derivation process. Despite its generality and utility, the adoption of this geometric language remains relatively obscure in applied fluid mechanics and is not widely recognized in the field of active nematics. In light of these challenges, our objective is to bridge this knowledge gap and recast a minimal active nematic system using the Lie derivative. This effort not only provides a unified perspective on the various components of the active nematohydrodynamic system but also, as we will demonstrate in the methods section, leads to a stable and elegant numerical approach [125, 74, 73].

Similar to the Lie derivative, we also seek to establish a coordinate-free framework for analyzing active nematic fields on curved surfaces by using the mathematical structure called *fiber bundles*. This structure encapsulates the concept of a family of tensor spaces parameterized by a base manifold, providing a coordinate-free geometric language for describing tensor fields on the manifold as *sections* of bundles. When each *fiber* in the bundle is a one-dimensional complex vector space equipped with a Riemannian metric, which houses the k -atic director and velocity vector, this bundle is considered a Hermitian complex line bundle. The theory of the complex line bundle has been the subject of extensive research in both pure mathematics and theoretical physics, particularly in modeling condensed matter phenomena [89]. The Levi-Civita connection on the tangent bundle induces an associated connection onto the nematic

bundle, which gives rise to a nematic Bochner Laplacian that accommodates fractional topological defects under the relaxation of nematic elasticity. The nuanced difference between the nematic and vector Bochner Laplacians has often been overlooked or inadequately explored in the existing literature. In addition to the theoretical advancement it provides, this complex line representation also facilitates the construction of an intrinsic complex-valued discrete Laplacian matrix based on the local chart, instead of relying on a global parametrization or an $\mathbb{R}^3$ representation in the embedded coordinate. Overall, the formalism grounded in the complex line bundle theory unveils a richer perspective on the representation of nematic—and more broadly k -atic—fields, across arbitrary 2-dimensional manifolds.

We also show how one can formulate the Stokes equations through the Dolbeault operators in the complex line bundle theory. On a Riemannian manifold, the measurement of strain rate is determined by the symmetric part of the covariant derivative, commonly referred to as the Killing operator. With the complex structure, the covariant derivative can be orthogonally decomposed into the holomorphic Del and the anti-holomorphic $Del \bar{}$ operators, jointly known as the Dolbeault operators. We elucidate that, under the incompressibility condition, the Laplacian-like operator that determines the viscous stress on a Riemannian manifold is induced by the $Del \bar{}$ operator. In addition to the Bochner Laplacian, this operator encompasses contributions from the curvature of the manifold. This fact is acknowledged in the literature, despite the results having been derived from the perspective of Riemannian geometry [43, 20, 106]. While both derivations are equivalent in this specific case, we believe that the more general formulation based on complex line bundles can reveal new perspectives and potential for further theoretical or computational advancements.

In the computational realm, several efforts have previously been undertaken to solve active fluid dynamics on curved surfaces. These endeavors are founded on continuum formulations [82, 94, 17, 97] or on the Lattice Boltzmann method [128, 129].

In particular, we highlight the limitations of using the streamfunction formulation for incompressible fluid mechanics on manifolds with non-trivial topology without specialized modifications [83, 127, 96, 88, 117, 17]. However, one common challenge with velocity-pressure-based methods is the representation of tangent vectors. In our method, we represent both the velocity and director using an intrinsic complex-valued function alongside the Levi-Civita connection. This approach eliminates the need for the commonly used numerical technique of constraining the $\mathbb{R}^3$ -valued velocity to lie tangent to the manifold [17, 48]. Moreover, we draw parallels between the numerical scheme for active nematics and classic problems in computational geometry. For example, insightful analogies can be found from the resemblance between the Stokes equations and the computation of Killing vector fields, which aid in discerning symmetries within geometries [11, 29, 27, 60]. Similarly, there is a correspondence between the modeling of k -atic liquid crystals with vector field design, which has been developed in applications such as mesh and texture generation [101, 60, 28].

The rest of the paper is structured as follows. We first introduce the geometric formulation of active nematics in a continuous setting in Sec. 3.2. This is followed by the development of a discretization approach that adheres to the underlying geometric principles in Sec. 3.3. Results are discussed in Sec. 3.4, where we present numerical solutions for the dynamics of active nematics on diverse geometries, highlighting the ability of our framework to capture the intricate dynamics under various flow regimes and surface topologies. We conclude and discuss potential extensions of this work in Sec. 3.5.

3.2 Theoretical formulation

An active fluid is a type of fluid whose constituent elements have the ability to exert internal mechanical stresses. In biological systems composed of microscopic

particles, inertia is negligible due to the small size and low velocities, resulting in low-Reynolds-number flow (i.e., Stokes flow). Here, the fluid flow is driven by the active stresses induced by the constituent elements. In active nematics, microscopic constituents with orientation $\mathbf{q}$ exhibit nematic symmetry, meaning there is an equivalence relation between orientations $\mathbf{q}$ and $-\mathbf{q}$. The configuration of the nematic field is governed by the nematodynamics equation, an advection-diffusion equation acting on this orientation. Because the system operates on a curved Riemannian surface and our discretization method is devoid of global coordinates, we first introduce the system of equations through an abstract geometric language and then present the explicit expression in a later section.

3.2.1 Mathematical framework

In this section, we will establish the geometric framework required to formulate the hydrodynamics of active nematics on a 2D Riemannian manifold. In Secs. 3.2.1 and 3.2.1, we introduce the complex representation of a k -atic director, specifically for the tangent vector and the nematic director. In Sec. 3.2.1, we introduce the Lie derivative on tensor fields, which is crucial for describing transport phenomena and deformations in fluid mechanics. The Bochner Laplacian governs the relaxation dynamics of vector and nematic fields. We start by introducing the Levi-Civita connection (i.e., covariant derivative) on the tangent bundle in Sec. 3.2.1, which enables us to extend the discussion to the nematic connection in Sec. 3.2.1 and the Bochner Laplacian on each bundle in Sec. 3.2.1. In Sec. 3.2.1, we detect and quantify charges associated with nematic topological defects by measuring the total rotation of a director field along a closed loop using the Levi-Civita connection.

Order parameter of k -atic director

We first introduce the general description for a k -atic director in the plane, such as a vector ($k = 1$) and a nematic director ($k = 2$). Let V be a 2D Euclidean vector space, and the multiplicative group action of complex numbers, $\mathbb{C}^\times$, on V is given by:

$$\mathbb{C}^\times \times V \rightarrow V, \quad (re^{i\theta}, \mathbf{u}) \mapsto \text{Scale}(\text{Rotate}(\mathbf{u}, \theta), r). \quad (3.1)$$

A finite subgroup of rotation actions, $N = \{1, e^{i2\pi/k}, \dots, e^{i(k-1)2\pi/k}\} \subset \mathbb{C}^\times$, defines an equivalence relation of k -rotational symmetry denoted by $\sim$. For any $\mathbf{u}, \mathbf{v} \in V$, we have:

$$\mathbf{u} \sim \mathbf{v} \quad \text{iff} \quad \exists \quad \hat{z} \in N \quad \text{s.t.} \quad \mathbf{v} = \hat{z}\mathbf{u}. \quad (3.2)$$

This equivalence relation allows us to define a quotient set $V_k = V/N$ and the map of taking the equivalence class $[\cdot]_k : V \rightarrow V_k$. Each element $[\mathbf{q}]_k \in V_k$ is called a k -atic director. The action of $\mathbb{C}^\times$ on V naturally induces a canonical quotient action $\mathbb{C}_k^\times = \mathbb{C}^\times / N$ on V_k , given by:

$$\mathbb{C}_k^\times \times V_k \rightarrow V_k, \quad (r^k e^{ik\theta}, [\mathbf{u}]_k) \mapsto [re^{i\theta}\mathbf{u}]_k, \quad (3.3)$$

where the rotational group is partitioned in θ , with $e^{ik\theta} \sim \hat{z}e^{ik\theta}, \hat{z} \in N$. Formally, we have the power map $P : \mathbb{C}^\times \rightarrow \mathbb{C}^\times, re^{i\theta} \mapsto r^k e^{ik\theta}$, which is a homomorphism with kernel being exactly N . By the first isomorphism theorem [53], $\mathbb{C}^\times / \ker(P) \cong \text{im}(P)$, the quotient group is isomorphic to the image of the map, $\mathbb{C}_k^\times \cong \text{im}(P) = \mathbb{C}^\times$. If given a unit basis vector $\mathbf{1} \in V$ with $|\mathbf{1}| = 1$, we can establish an isomorphism between complex numbers $\mathbb{C} \ni \hat{q}$ and vectors $V \ni \mathbf{q}$ by $\mathbf{q} = \hat{q}\mathbf{1} = re^{i\theta}\mathbf{1} \in V$. Similarly, given a k -director basis $[\mathbf{1}]_k \in V_k$, we can establish the isomorphism between $\mathbb{C} \ni [\hat{q}]_k$ and its associated k -director $V_k \ni [\mathbf{q}]_k = [\hat{q}]_k[\mathbf{1}]_k = r^k e^{ik\theta}[\mathbf{1}]_k = \hat{q}^k[\mathbf{1}]_k$. Through the chain rule,

the *pushforward* of the equivalence map, denoted as $d[\cdot]_k|_q$, maps an increment vector, $q' \in T_q V$, to an increment k -atic director, $[q]_k' = k[q]_{k-1} q' \in T_{[q]_k} V_k$.

When focusing on a nematic field with $k = 2$ and $N = \{1, -1\}$, an alternative representation of the equivalence class is through a rank-1 symmetric tensor $[q]_2 \cong q \otimes q \in \mathcal{Q}$, where $\mathcal{Q} = \{ Q \in V \otimes_{\text{symm}} V \mid \text{rank } Q \leq 1 \}$. We called the bijective map from the representation of complex numbers to the matrix representation the *Veronese map*,

$$\mathcal{V} : V_2 \rightarrow \mathcal{Q}, \quad [q]_2 \mapsto q \otimes q. \quad (3.4)$$

For clarity, in the reminder of this paper, we will omit the subscript and specify the bracket to denote a nematic equivalence (i.e., $[\cdot] = [\cdot]_2$). With $[q] \in V_2$, $Q = \mathcal{V}([q]) \in \mathcal{Q}$, the pushforward of the Veronese map $d\mathcal{V} : T_{[q]} V_2 \rightarrow T_Q \mathcal{Q}$ maps an increment $[q]'$ in complex representation to the increment in matrix representation Q' .

Vector field and nematic field on a Riemannian manifold

On a closed Riemannian 2-manifold M , $\partial M = \emptyset$, each point $p \in M$ has a tangent space $T_p M$ that is a copy of the Euclidean plane: $T_p M \cong V$. Tangent spaces at different points, $T_p M$, $T_q M$, for $p \neq q$, are disjoint spaces (spaces with no elements in common), and the disjoint union (a union where elements are kept separate rather than merged) of all tangent spaces is referred to as the *tangent bundle*, $TM = \bigcup_p T_p M$. Here the tangent space $T_p M$ is also referred to as the *fiber* of the *fiber bundle* TM at a base point $p \in M$. A tangent vector field q is described as a *section* of the tangent bundle, $q \in \Gamma(TM)$, which is an assignment of a fiber element $q_p \in T_p M$ at each point $p \in M$. We also denote a section defined over a region $U \subset M$ as $q \in \Gamma_U(TM)$.

Similarly, we can describe a nematic field as a section of the nematic bundle L , $[q] \in \Gamma(L)$. Each fiber of the nematic bundle L at $p \in M$, L_p , is given by the equivalence

relation, $L_p := [T_p M](= [T_p M]_2)$ (c.f. Sec 3.2.1). In other words, the nematic bundle $L = [TM]$ is obtained by taking fiberwise equivalence class of the tangent bundle.

With a basis section for the tangent bundle, $\mathbf{1} \in \Gamma_U(TM)$, $|\mathbf{1}| = 1$, defined in a neighborhood $U \subset M$, we can construct the corresponding nematic basis section for L as $[\mathbf{1}] \in \Gamma_U(L)$. Basis sections allow us to represent vector fields and nematic fields by complex scalar functions, $\hat{q}: U \rightarrow \mathbb{C}$ and $[\hat{q}]: U \rightarrow \mathbb{C}_2$.

Recall that a nematic vector $[q]$ can also be represented by a rank-1 symmetric tensor via the Veronese map of Eq. (3.4). We can similarly represent a nematic field as a section of the symmetric rank-1 tensor bundle as follows. The complex representation of the nematic field is isomorphic to a tensor field $\mathcal{Q} = q \otimes q \in \Gamma(E)$, where E is the fiber bundle given by the set $\mathcal{Q}$ of rank ≤ 1 in the symmetric tensor bundle $TM \otimes_{\text{symm}} TM$. By populating the fiber-wise Veronese map $\mathcal{V}: V_2 \rightarrow \mathcal{Q}$, we get the section-wise map $\mathcal{V}: \Gamma(L) \rightarrow \Gamma(E)$.

The Riemannian metric structure, $g|_p: T_p M \times_{\text{symm}} T_p M \rightarrow \mathbb{R}$, provides a positive-definite inner product $\langle \cdot, \cdot \rangle = g(\cdot, \cdot)$ between tangent vector fields or nematic fields. For any two vector fields $u = \hat{u}\mathbf{1}$, $v = \hat{v}\mathbf{1}$ and two nematic fields $[q] = [\hat{q}][\mathbf{1}] \cong q \otimes q$, $[p] = [\hat{p}][\mathbf{1}] \cong p \otimes p$, the inner products are defined using their complex representations,

$$\langle \cdot, \cdot \rangle : \begin{cases} V \times V \rightarrow \mathbb{R}, & \langle u, v \rangle \mapsto \Re(\hat{u}\bar{\hat{v}}) \\ V_2 \times V_2 \rightarrow \mathbb{R}, & \langle [q], [p] \rangle \mapsto \Re([\hat{q}]\bar{[\hat{p}]}) \\ \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}, & \langle p \otimes p, q \otimes q \rangle = \langle p, q \rangle^2 \mapsto (\Re(\hat{q}\bar{\hat{p}}))^2, \end{cases} \quad (3.5)$$

where $\bar{\cdot}$ denotes complex conjugation. Note that the inner product for $\mathcal{Q}$ is the Frobenius inner product that can be extended and denoted as $G = g \otimes g: (V \otimes V) \times (V \otimes V) \rightarrow \mathbb{R}$. A functional that maps a tangent vector field to a scalar function, $\alpha: T_p M \rightarrow \mathbb{R}$, is referred to as a *covector* field. A natural *dual pairing* of α with a vector field u is denoted

as $\alpha[\mathbf{u}]$. We denote this as $\alpha \in \Gamma(T^*M)$, where T^*M is known as the cotangent bundle, which is the dual space to the tangent bundle. The metric provides a way to associate a vector with its dual counterpart, $g|_p : T_pM \rightarrow T_p^*M$, where a vector $\mathbf{u}$ is mapped to $g(\mathbf{u}, \cdot)$. In index notation, this is referred to as *lowering* an index, and it can be expressed as $u_i = g_{ij}u^j$. Similarly, the Frobenius inner product $G|_p : (T_pM \otimes T_pM) \rightarrow (T_p^*M \otimes T_p^*M)$ provides a way to map a $(2,0)$ tensor to a $(0,2)$ tensor, or $P_{ij} = g_{im}g_{jn}P^{mn}$ in index notation. With the metric, we can define the L_2 inner product of a vector or nematic field on M as $\langle\langle \cdot, \cdot \rangle\rangle := \int_M \langle \cdot, \cdot \rangle dA : \Gamma(\circ) \times \Gamma(\circ) \rightarrow \mathbb{R}$. The corresponding metric norm and L_2 norm are denoted as $|\cdot|$ and $\|\cdot\|$.

Flow, Lie derivative, and advection

On a manifold, the Lie derivative provides a generalized framework for assessing changes in tensor fields encompassing functions with scalar, vector, and covector values, along the flow defined by another vector field [66, 107, 56].

Consider an instantaneous flowmap $\varphi(t) : M \rightarrow M$ generated by a velocity field $\mathbf{u} \in \Gamma(TM)$ such that $\dot{\varphi} = \mathbf{u}$, $\varphi(0) = \text{id}_M$. The flowmap φ induces a pushforward map on the tangent space, $d\varphi|_p : T_pM \rightarrow T_{\varphi(p)}M$. In continuum mechanics, the pushforward map is known as the deformation gradient $\mathbf{F}$ and satisfies $d\varphi|_p(\mathbf{q}) = \mathbf{F}\mathbf{q}$. We can deduce from linearity that $d\varphi([\mathbf{q}]) = [d\varphi(\mathbf{q})] = [\mathbf{F}\mathbf{q}]$. This is equivalent to the map on the matrix representation $\mathbf{Q}$, $d\varphi(\mathbf{q} \otimes \mathbf{q}) = d\varphi(\mathbf{q}) \otimes d\varphi(\mathbf{q}) = \mathbf{F}\mathbf{Q}\mathbf{F}^\top$. To summarize, we have the pushforward map defined as

$$d\varphi : \begin{cases} \Gamma(TM) \rightarrow \Gamma(TM), & d\varphi(\mathbf{q}) = \mathbf{F}\mathbf{q} \\ \Gamma(L) \rightarrow \Gamma(L), & d\varphi([\mathbf{q}]) = [\mathbf{F}\mathbf{q}] \\ \Gamma(E) \rightarrow \Gamma(E), & d\varphi(\mathbf{Q}) = \mathbf{F}\mathbf{Q}\mathbf{F}^\top. \end{cases} \quad (3.6)$$

For a covector field $\alpha \in \Gamma(T^*M)$, the pullback map $\varphi^* : T_{\varphi(p)}^*M \rightarrow T_p^*M$, $\varphi^*\alpha \mapsto$

$(d\varphi)^\top \boldsymbol{\alpha} = \mathbf{F}^\top \boldsymbol{\alpha}$, is the adjoint of the pushforward map. Because the metric $\mathbf{g} \in \Gamma(T^*M \otimes_{\text{symm}} T^*M)$ is a valence $(0,2)$ tensor, its pullback map follows $\varphi^* \mathbf{g} \mapsto \mathbf{F}^\top \mathbf{g} \mathbf{F}$. Since the pullback map preserves the dual pairing, $\varphi^*(\boldsymbol{\alpha}[\mathbf{q}]) = \varphi^* \boldsymbol{\alpha}[\varphi^* \mathbf{q}]$, a pullback map for a vector field is the inverse map of the pushforward $\varphi^*|_p = (d\varphi|_p)^{-1} = \mathbf{F}^{-1} : T_{\varphi(p)}M \rightarrow T_pM$. With omission of the precomposition for non-scalar fields (e.g., $\mathbf{q} \circ \varphi$ is denoted as $\mathbf{q}$), we can summarize pullback maps as

$$\varphi^* : \begin{cases} C^\infty(M) \rightarrow C^\infty(M), & \varphi^* f = f \circ \varphi \\ \Gamma(G) \rightarrow \Gamma(G), & \varphi^* \mathbf{g} = \mathbf{F}^\top \mathbf{g} \mathbf{F} \\ \Gamma(TM) \rightarrow \Gamma(TM), & \varphi^* \mathbf{q} = \mathbf{F}^{-1} \mathbf{q} \\ \Gamma(L) \rightarrow \Gamma(L), & \varphi^* [\mathbf{q}] = [\mathbf{F}^{-1} \mathbf{q}] \\ \Gamma(E) \rightarrow \Gamma(E), & \varphi^* \mathbf{Q} = \mathbf{F}^{-1} \mathbf{Q} \mathbf{F}^{-\top}. \end{cases} \quad (3.7)$$

The Lie derivative is defined as the rate of change of the time-varying pullback field at $t = 0$,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} \circ \varphi^* =: \frac{\partial}{\partial t} + \mathcal{L}_{\dot{\varphi}} : \begin{cases} \dot{f} + \mathcal{L}_u f & = \lim_{t \rightarrow 0} (f_{\varphi(p)} - f_p) / t \\ \mathcal{L}_u \mathbf{g} & = \lim_{t \rightarrow 0} (\mathbf{F}^\top \mathbf{g}_{\varphi(p)} \mathbf{F} - \mathbf{g}_p) / t \\ \dot{\mathbf{q}} + \mathcal{L}_u \mathbf{q} & = \lim_{t \rightarrow 0} (\mathbf{F}^{-1} \mathbf{q}_{\varphi(p)} - \mathbf{q}_p) / t \\ [\dot{\mathbf{q}}] + \mathcal{L}_u [\mathbf{q}] & = \lim_{t \rightarrow 0} ([\mathbf{F}^{-1} \mathbf{q}_{\varphi(p)}] - [\mathbf{q}]_p) / t \\ \dot{\mathbf{Q}} + \mathcal{L}_u \mathbf{Q} & = \lim_{t \rightarrow 0} (\mathbf{F}^{-1} \mathbf{Q}_{\varphi(p)} \mathbf{F}^{-\top} - \mathbf{Q}_p) / t. \end{cases} \quad (3.8)$$

Here, we assume that the metric is time-independent, $\dot{\mathbf{g}} = 0$. In continuum mechanics, the Lie derivative of scalar function recovers the familiar material derivative. When applied to a vector and nematic field, the Lie derivative is the upper-convected derivative [86]. The Lie derivative of the metric, $\mathcal{L}_u \mathbf{g}$, is the strain rate tensor that measures the

rate of change of the Green-Lagrange strain tensor, $F^\top \mathbf{g} F - \mathbf{g}$. A section $\boldsymbol{\psi}$ is said to be *Lie advected* by the flow φ when $\partial \varphi^* \boldsymbol{\psi} / \partial t = 0$.

In summary, the Lie derivative generalizes the concept of advection from scalar functions to tensor fields. This theoretical framework is not only mathematically satisfying but also practical, as it enables the development of a unifying geometric discretization (cf. Sec. 3.3.2) that forgoes the use of coordinate-based expressions and respects the underlying continuous structure.

Levi-Civita connection on the tangent bundle

On a curved space, each tangent space is in general different from other tangent spaces. The Levi-Civita connection introduces the concept of parallel transport, allowing us to compare vectors at different points along a curved manifold. Given a parametrized segment on M , $\gamma(t) : [-\epsilon, \epsilon] \rightarrow M$, with the starting point $\gamma(0) = p$ and velocity $\gamma'(0) = v$, parallel transport is a one-parameter family of linear maps dependent on the path γ , denoted as $\Pi_{p \rightsquigarrow \gamma(t)} : T_p M \rightarrow T_{\gamma(t)} M$, with $t \in [-\epsilon, \epsilon]$ [78]. Parallel transport allows us to express the directional covariant derivative of $\mathbf{u} \in \Gamma(TM)$ with respect to v as follows:

$$\nabla_v \mathbf{u} = \left. \frac{\nabla}{dt} \mathbf{u}(\gamma(t)) \right|_{t=0} := \lim_{t \rightarrow 0} \frac{\Pi_{\gamma(t) \rightsquigarrow p} \mathbf{u}(\gamma(t)) - \mathbf{u}(p)}{t}, \quad (3.9)$$

where $\Pi_{\gamma(\epsilon) \rightsquigarrow p}$ denotes the inverse of $\Pi_{p \rightsquigarrow \gamma(\epsilon)}$. The covariant derivative, $\nabla : \Gamma(TM) \rightarrow \Gamma(TM \otimes T^*M)$, is defined by the relation $\nabla \mathbf{u} \llbracket v \rrbracket := \nabla_v \mathbf{u}$ or, in index notation, as $v^i \nabla_i u^j$.

Recall that under a local basis section, $\mathbf{1} \in \Gamma_U(TM)$, where $|\mathbf{1}| = 1$, a tangent vector field $\mathbf{u} \in \Gamma_U(TM)$ can be represented as a complex field $\hat{u} : U \rightarrow \mathbb{C}$, given by $\mathbf{u} = \hat{u} \mathbf{1}$. In this complex representation, the covariant derivative can be expressed as

$$\nabla \mathbf{u} = (d\hat{u}) \mathbf{1} + \hat{u} (\nabla \mathbf{1}) =: ((d + \mathbf{i}\omega) \hat{u}) \mathbf{1}. \quad (3.10)$$

where $d\hat{u} = d\Re(\hat{u}) + \mathbf{i}d\Im(\hat{u}) \in \Gamma_U(T^*M; \mathbb{C})$ represents a complex-valued covector field and ω is a real-valued covector field $\omega \in \Gamma_U(T^*M; \mathbb{R})$ defined by the covariant derivative of the basis field $\nabla \mathbf{1} =: \mathbf{i}\omega \mathbf{1} \in \Gamma_U(TM \otimes T^*M)$. The components of ω are commonly referred to as the Christoffel symbols.

Holomorphic structure with Levi-Civita connection

As being demonstrated repeatedly, a vector field, when equipped with a basis field, can be viewed as a complex scalar field. Using the Levi-Civita connection as the spatial derivative operator for this complex field, we can distinguish its holomorphic and anti-holomorphic components by orthogonally decomposing the covariant derivative into the *Del* and *Del bar* operators, $\nabla = \partial^\nabla + \bar{\partial}^\nabla$,

$$\begin{aligned}\bar{\partial}_v^\nabla \mathbf{u} &:= \frac{\nabla_v \mathbf{u} + \mathbf{i}\nabla_{iv} \mathbf{u}}{2} \\ \partial_v^\nabla \mathbf{u} &:= \frac{\nabla_v \mathbf{u} - \mathbf{i}\nabla_{iv} \mathbf{u}}{2}.\end{aligned}\tag{3.11}$$

Note that $\mathbf{i}$ represents rotation of the vector by 90° , $\mathbf{i}(\cdot) = \text{Rotate}(\cdot, 90^\circ)$. A vector field $\mathbf{u}$ is holomorphic if and only if $\nabla \mathbf{u}[\mathbf{i}\cdot] = \mathbf{i}\nabla \mathbf{u}[\cdot]$, or $\bar{\partial}^\nabla \mathbf{u} = 0$. Its conjugate $\partial^\nabla \mathbf{u} = 0$ indicates $\mathbf{u}$ is antiholomorphic.

It turns out that the holomorphic and anti-holomorphic parts play important roles when the vector field $\mathbf{u}$ represents the velocity field of a surface flow. When $\mathbf{u}$ denotes the fluid velocity, the holomorphic part, $\partial^\nabla \mathbf{u}$, quantifies the extent of conformal deformation (involving infinitesimal rotation with local dilation) of the fluid. The anti-holomorphic part, $\bar{\partial}^\nabla \mathbf{u}$, corresponds to the symmetric and trace-free part of $\nabla \mathbf{u}$. It is the strain rate tensor for incompressible flow and measures the infinitesimal shearing that viscosity resists. This concept will be applied in Sec. 3.2.2 when formulating the Stokes equations on the manifold.

Levi-Civita connection on the nematic bundle

We can extend the Levi-Civita connection from the tangent bundle to the nematic bundle. This is equivalent to defining a nematic covariant derivative, denoted by $\nabla^L : \Gamma(L) \rightarrow \Gamma(L \otimes T^*M)$. Under a local nematic basis $[\mathbf{1}] \in \Gamma_U(L)$, the nematic covariant derivative can be defined as:

$$\nabla^L[\mathbf{q}] = d[\hat{q}]\mathbf{1} + [\hat{q}]\nabla^L[\mathbf{1}] := \left((d + \mathbf{i}2\omega)[\hat{q}] \right) [\mathbf{1}], \quad (3.12)$$

where ω is the covector field $\mathbf{i}\omega\mathbf{1} := \nabla\mathbf{1}$ associated to the tangent bundle connection and basis. The nematic connection $\nabla^L[\mathbf{1}] := \mathbf{i}2\omega[\mathbf{1}]$ reflects increased rotational speed due to nematic symmetry. This definition is basis-independent and aligns with the algebraic construction of the Levi-Civita connection on the tangent bundle, $\nabla^L[\mathbf{q}] = 2\mathbf{q}\nabla\mathbf{q} = \nabla(\mathbf{q}^2)$.

The nematic covariant derivative $\nabla^L[\mathbf{q}]$ is consistent with the standard covariant derivative for matrix-valued functions, $\nabla : \Gamma(TM \otimes TM) \rightarrow \Gamma(TM \otimes TM \otimes T^*M)$, when applied to $\mathbf{Q} = \mathcal{V}([\mathbf{q}]) \in \Gamma(E) \subset \Gamma(TM \otimes TM)$. Via the pushforward of the Veronese map, $d\mathcal{V}|_{[\mathbf{q}]} : T_{[\mathbf{q}]} \Gamma(L) \rightarrow T_{\mathcal{V}([\mathbf{q}])} \Gamma(E)$, one can confirm the equivalence of these two derivatives, $\nabla\mathbf{Q} = \nabla\mathbf{q} \otimes \mathbf{q} + \mathbf{q} \otimes \nabla\mathbf{q} = d\mathcal{V}|_{[\mathbf{q}]}[\nabla^L[\mathbf{q}]]$.

Laplacian and diffusion

The concept of the covariant derivative naturally leads to a Laplacian operator referred to as the Bochner Laplacian, which is the first variation of the Dirichlet energy.

To begin, we introduce the Bochner Laplacian when it operates on vector fields. The Dirichlet energy functional maps a vector field to an energy measure, $\mathcal{F} : \Gamma(TM) \rightarrow \mathbb{R}$, and can be expressed as follows:

$$\mathcal{F}(\mathbf{u}) = \frac{1}{2} \|\nabla\mathbf{u}\|^2 \stackrel{\partial M = \emptyset}{=} \frac{1}{2} \langle\langle \mathbf{u}, \nabla^* \nabla \mathbf{u} \rangle\rangle =: -\frac{1}{2} \langle\langle \mathbf{u}, \Delta \mathbf{u} \rangle\rangle, \quad (3.13)$$

where the operator $\nabla^* : \Gamma(TM \otimes T^*M) \rightarrow \Gamma(TM)$ represents the L_2 adjoint of ∇ , defining the Bochner Laplacian as $\Delta := -\nabla^*\nabla : \Gamma(TM) \rightarrow \Gamma(TM)$. Note that we use the negative semi-definite convention for the Laplacian operator.

Using the same procedure, the nematic covariant derivative is employed to construct the nematic Dirichlet energy, $\mathcal{F}^L \in \Gamma(L) \rightarrow \mathbb{R}$, which is expressed as:

$$\mathcal{F}^L([q]) = \frac{1}{2} \|\nabla^L[q]\|^2 \stackrel{\partial M = \emptyset}{=} \frac{1}{2} \langle\langle [q], \nabla^{L*} \nabla^L[q] \rangle\rangle =: -\frac{1}{2} \langle\langle [q], \Delta^L[q] \rangle\rangle. \quad (3.14)$$

The variation of $\mathcal{F}^L$ leads to the definition of the nematic Bochner Laplacian, denoted as $\Delta^L := -\nabla^{L*} \nabla^L : \Gamma(L) \rightarrow \Gamma(L)$.

Analogously, the standard Bochner Laplacian $\Delta : \Gamma(TM \otimes TM) \rightarrow \Gamma(TM \otimes TM)$ defined for a general matrix field $\mathbf{H}$ can be derived from the matrix Dirichlet energy, $\mathcal{E} : \Gamma(TM \otimes TM) \rightarrow \mathbb{R}$, measured by the Frobenius inner product, $\mathcal{E}(\mathbf{H}) = \frac{1}{2} \|\nabla \mathbf{H}\|^2 = \frac{1}{2} \int_M \nabla_i H_{jk} \nabla^i H^{jk} dA$. This Laplacian Δ can act on a nematic field $\mathbf{Q} = \mathcal{V}([q])$, but it differs from the nematic Bochner Laplacian, $\Delta \mathbf{Q} \neq d\mathcal{V}|_{[q]} \llbracket \Delta^L[q] \rrbracket$.

The reason for this difference is two-fold. First, nematic fields only correspond to rank-1 matrices in the nematic subspace $E \subset TM \otimes TM$, but the matrix Bochner Laplacian spans the entire $TM \otimes TM$. This results in the matrix representation of nematic fields being an overparametrization under this operator. This overparametrization can be demonstrated by considering the matrix diffusion equation $\dot{\mathbf{Q}} = \Delta \mathbf{Q}$. In general, a rank-1 initial condition $\mathbf{Q}|_{t=0} \in \Gamma(E)$ does not remain rank-1 under this diffusion (i.e., $\mathbf{Q}$ diverges from the nematic subspace E). In contrast, $\dot{[q]} = \Delta^L[q]$ remains within the nematic phase. Second, despite the equivalence of covariant derivatives $\nabla \mathbf{Q} = d\mathcal{V}|_{[q]} \llbracket \nabla^L[q] \rrbracket$ (cf. Sec. 3.2.1), their corresponding Dirichlet energies are distinct: $\mathcal{E}(\mathbf{Q}) \neq \mathcal{F}^L([q])$. This is because the pushforward of the Veronese map $d\mathcal{V}|_{[q]} : T_{[q]}L \rightarrow T_{\mathbf{Q}}E$ does not isometrically map the intrinsic norm on $T_{[q]}L$ to the Frobenius norm for matrices on $T_{\mathbf{Q}}E$.

As a side remark, it is possible to represent the nematic Dirichlet energy $\mathcal{F}^L([q])$ in terms of Q through an equivalent functional $\mathcal{F}^E: \Gamma(E) \rightarrow \mathbb{R}$ such that $\mathcal{F}^E(Q) = \mathcal{F}^L([q])$. We can achieve this by introducing a specific quadratic form, $|\cdot|_{\mathcal{Q}}^2 := |\cdot|^2 - 2 \det(\cdot)$, for $\mathbb{R}^{2 \times 2}$ matrices on $TM \otimes TM$. This quadratic form induces a corresponding symmetric bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{Q}} = \langle \cdot, \cdot \rangle - 2 \det(\cdot, \cdot)$. The equivalent nematic Dirichlet energy $\mathcal{F}^E$ relies on this bilinear form and can be explicitly expressed in index notation:

$$\mathcal{F}^E(Q) = \frac{1}{2} \int_M |\nabla Q|_{\mathcal{Q}}^2 dA \equiv \frac{1}{2} \int_M g^{ij} (g_{km} g_{\ell n} - \det(g) \epsilon_{km} \epsilon_{\ell n}) \nabla_i Q^{k\ell} \nabla_j Q^{mn} dA. \quad (3.15)$$

Here, g_{ij} are the components of the metric tensor g and ϵ_{ij} represents the Levi-Civita permutation symbol.

In this paper, we adopt the more natural choices of $\mathcal{F}^L([q])$ and Δ^L to model relaxation of the nematic field. We use $\Delta^E: \Gamma(E) \rightarrow \Gamma(E)$ to denote the Laplacian equivalent to Δ^L but acting on the matrix field representation. That is, Δ^E satisfies the equivalence $\Delta^E Q = \Delta^E(\mathcal{V}[q]) = d\mathcal{V}[\Delta^L[q]] \in T_Q \Gamma(E)$ and is defined as,

$$\Delta^E Q := -\frac{\delta \mathcal{F}^E(Q)}{\delta Q|_E} = -d\mathcal{V} \left[\frac{\delta \mathcal{F}([q])}{\delta [q]} \right] = d\mathcal{V} \circ \Delta^L \circ \mathcal{V}^{-1}(Q). \quad (3.16)$$

The subscript $|_E$ emphasizes the variation within the nematic subspace E , rather than the entire $TM \otimes TM$.

Curvature, defects, and local-global geometry

The singularities within the nematic field, also referred to as topological defects, are associated with points where the director field exhibits zeros and discontinuities. These defects are commonly characterized by their index or charge $\mathcal{Z}$. To identify these defects and compute their charges, we employ the covariant derivative in closed-loop integration, a classic procedure also discussed in other texts [78, 79].

The charge of a topological defect in a k -atic field $[q]_k$ is determined by the

cumulative turning angle that the director field undergoes around the defect. Here we consider a defect at point $p \in U$, where $U \subset M$ represents a sufficiently small disk that does not encircle any other singularities. The boundary of the disk is denoted as $\gamma = \partial U$. Locally in U it is always possible to establish a smooth and defect-free basis $[\mathbf{1}]_k$ and subsequently express the rotation form under the Levi-Civita connection, denoted as $\eta \in \Gamma(T^*M)$, as

$$\eta := \frac{1}{|[\mathbf{q}]_k|^2} \langle \nabla[\mathbf{q}]_k, \mathbf{i}[\mathbf{q}]_k \rangle = \frac{1}{|[\mathbf{q}]_k|^2} \langle d[\mathbf{q}]_k + \mathbf{i}k\omega[\mathbf{q}]_k, \mathbf{i}[\mathbf{q}]_k \rangle = d^{\mathbb{S}^1} \arg([\hat{q}]_k) + k\omega. \quad (3.17)$$

Note that the function $\arg$ necessitates a smooth basis to be well-behaved and returns values in $\mathbb{S}^1$ rather than real numbers. Despite being $\mathbb{S}^1$ -valued by itself, the differential, $d^{\mathbb{S}^1} \arg([\hat{q}]_k)$, can be defined as a real-valued function. This is because, at a given point, this differential depends only on the function's values in a small neighborhood around the point. In such a local context, a continuous $\mathbb{S}^1$ -valued function can be unwrapped to a real-valued function up to a 2π -integer constant that does not affect the differential.

The closed-loop integral of the turning rate η defined in Eqn. (3.17) along the parametrized boundary loop $\gamma(t) = \partial U(t)$, where $t \in [0, T]$ and $\gamma(0) = \gamma(T)$, is referred to as the cumulative turning angle of the director field around a defect. The integral of the connection ω in Eqn. (3.17), which is known as the holonomy angle, is equal to the negative of the integrated Gaussian curvature: $\oint_{\gamma} \omega = - \int_U K dA$. Therefore, the charge of the topological defect in a k -atic director field, denoted as $\mathcal{Z} \in \mathbb{Z}/k$, admits the following relation:

$$\mathcal{Z}_p([\mathbf{q}]_k) := \frac{1}{2\pi k} \oint_{\gamma} d^{\mathbb{S}^1} \arg([\hat{q}]_k) = \frac{1}{2\pi k} \oint_{\gamma} \eta + \frac{1}{2\pi} \int_U K dA. \quad (3.18)$$

Note that the closed integral ensures that the orientation after integration, $\arg([\hat{q}]_k)|_{t=T}$, differs only by an integer multiple of 2π from its starting value, $\arg([\hat{q}]_k)|_{t=0}$. As a

result, the charge is consistently an integer multiple of $1/k$ (i.e., an integer value for a vector field and a multiple of $1/2$ for a nematic field). This quantized property also implies that the charge of the defect remains invariant regardless of variations in the size and shape of γ and is solely the property of the director field. As a special case, $\oint_{\gamma} dS^1 \arg([\hat{q}]_k) = 0$ if γ does not enclose any topological defects of $\arg([\hat{q}]_k)$.

Owing to the Poincaré-Hopf Theorem, we can relax the local constraint on U . This theorem relates the sum of charges on the entire manifold M (with $\partial M = \emptyset$) to the topological invariant χ known as the Euler characteristic:

$$\sum_i \mathcal{Z}_{p_i}([q]_k) = \chi(M) = 2 - 2g, \quad (3.19)$$

where g represents the genus of the manifold. By replacing U with manifold M , Eqn. (3.18) leads to the Gauss-Bonnet Theorem, which states that the total Gaussian curvature is also a topological invariant, $\int_M K dA = 2\pi\chi(M)$.

In a nutshell, we can detect defects and calculate their charges by establishing the cumulative turning angle through the use of the covariant derivative. Local-to-global theorems, including the Gauss-Bonnet and Poincaré-Hopf theorems, connect local properties like curvature of the surface and charges of the nematic field to robust global invariants. Practically, as we will elaborate in Sec. 3.3.5, these formulas translate to the discretization of Gaussian curvature and the algorithmic computation of topological charges, which satisfy the discrete analog of local-to-global theorems.

Expressing the Lie derivative as a covariant derivative

Under the Levi-Civita connection, the Lie derivative can be expressed in terms of the covariant derivative. To demonstrate this, we denote the directional derivative, a derivation associated with its corresponding vector field, as $u := d_u : C^\infty(M) \rightarrow C^\infty(M)$. In index notation, it is commonly denoted as $u^i \partial_i$. The pullback map respects

the derivation structure; for $\mathbf{q} \in \Gamma(TM)$, $f \in C^\infty(M)$, we have $\varphi^*(d_{\mathbf{q}}f) = d_{\varphi^*\mathbf{q}}\varphi^*f$. Consequently, the Lie derivative satisfies the Leibniz rule over derivation, $\mathcal{L}_{\mathbf{u}}(d_{\mathbf{q}}f) = d_{\mathcal{L}_{\mathbf{u}}\mathbf{q}}f + d_{\mathbf{q}}\mathcal{L}_{\mathbf{u}}f$. Since the Lie derivative of a scalar field is equivalent to the directional derivative, we have $\mathcal{L}_{\mathbf{u}}\mathbf{q}(f) := d_{\mathcal{L}_{\mathbf{u}}\mathbf{q}}f = \mathcal{L}_{\mathbf{u}}(d_{\mathbf{q}}f) - d_{\mathbf{q}}\mathcal{L}_{\mathbf{u}}f = \mathbf{u}\mathbf{q}(f) - \mathbf{q}\mathbf{u}(f) =: [\mathbf{u}, \mathbf{q}](f)$, known as the Lie bracket. The Lie bracket can in fact be expressed using the covariant derivative of the Levi-Civita connection, referred to as the torsion-free condition of the connection, $T^\nabla[[\mathbf{u}, \mathbf{q}]] := \nabla_{\mathbf{u}}\mathbf{q} - \nabla_{\mathbf{q}}\mathbf{u} - [\mathbf{u}, \mathbf{q}] = 0$. Under the Levi-Civita connection, the Lie material derivative for vector $\mathbf{q}$ can be expressed as

$$\dot{\mathbf{q}} + \mathcal{L}_{\mathbf{u}}\mathbf{q} = \dot{\mathbf{q}} + \nabla_{\mathbf{u}}\mathbf{q} - \nabla_{\mathbf{q}}\mathbf{u}, \quad (3.20)$$

which is a familiar form of the upper-convected derivative, or Oldroyd derivative for vectors [86]. The term $\nabla_{\mathbf{u}}\mathbf{q}$ represents the parallel transport components, while $\nabla_{\mathbf{q}}\mathbf{u}$ accounts for the local deformation caused by the differential flow velocity.

Through the pushforward of k -atic equivalence $d[\cdot]_k$ (cf. Sec. 3.2.1), the Lie derivative of the k -atic director field can be expressed as follows:

$$[\dot{\mathbf{q}}]_k + \mathcal{L}_{\mathbf{u}}[\mathbf{q}]_k = k\mathbf{q}^{k-1}(\dot{\mathbf{q}} + \mathcal{L}_{\mathbf{u}}\mathbf{q}), \quad (3.21)$$

where $\mathbf{q}^{k-1}$ denotes the $(k-1)$ -th power of the nematic $\mathbf{q}$ as a complex function. When $k=2$, this is equivalent to the advection of the tensorial representation of the nematic field. Applying the Leibniz rule of the Lie derivative to the tensor product, we have $\mathcal{L}_{\mathbf{u}}\mathbf{Q} = \mathcal{L}_{\mathbf{u}}\mathbf{q} \otimes \mathbf{q} + \mathbf{q} \otimes \mathcal{L}_{\mathbf{u}}\mathbf{q}$. By rearranging the terms, we recover the operator for the upper-convected derivative of the symmetric tensor $\mathbf{Q}$,

$$\begin{aligned} \dot{\mathbf{Q}} + \mathcal{L}_{\mathbf{u}}\mathbf{Q} &= \dot{\mathbf{Q}} + \nabla_{\mathbf{u}}\mathbf{q} \otimes \mathbf{q} + \mathbf{q} \otimes \nabla_{\mathbf{u}}\mathbf{q} - \nabla_{\mathbf{q}}\mathbf{u} \otimes \mathbf{q} - \mathbf{q} \otimes \nabla_{\mathbf{q}}\mathbf{u} \\ &= \dot{\mathbf{Q}} + \nabla_{\mathbf{u}}\mathbf{Q} - (\nabla\mathbf{u})\mathbf{Q} - \mathbf{Q}(\nabla\mathbf{u}^\top). \end{aligned} \quad (3.22)$$

The dual pairing order follows the convention of matrix multiplication (i.e., $(\mathcal{L}_u Q)^{ij} = u^k \nabla_k Q^{ij} - \nabla_k u^i Q^{kj} - Q^{ik} \nabla_k u^j$ in index notation).

As demonstrated in Sec. 3.2.1, the strain rate tensor can be formulated as the Lie derivative of the metric, $\mathcal{L}_u g$. When expressing it in terms of the covariant derivative acting on the Eulerian velocity, we can retrieve its well-known form as the symmetric component of the velocity gradient. By expressing $\dot{F} = (d\dot{\varphi}) = \varphi^*(\nabla \dot{\varphi}) = (\nabla u)F \in \Gamma(TM \otimes T^*M)$, we can express the Lie derivative of the metric as the pullback of the strain rate tensor,

$$\mathcal{L}_u g = \frac{\partial}{\partial t}(F^\top g F) = \dot{F}^\top g F + F^\top g \dot{F} = F^\top (\nabla u^\top g + g \nabla u) F = \varphi^*(\nabla u^\top g + g \nabla u). \quad (3.23)$$

Since the flowmap, $\varphi : M \rightarrow M$, is instantaneously an identity map, we have $\mathcal{L}_u g = \nabla u^\top g + g \nabla u \in \Gamma(T^*M \otimes_{\text{symm}} T^*M)$, which measures the infinitesimal deformation rate of a fluid element. This Lie derivative defines the Killing operator,

$$\mathcal{K} : \Gamma(TM) \rightarrow \Gamma(T^*M \otimes_{\text{symm}} T^*M), \quad \mathcal{K} : u \mapsto \frac{1}{2} \mathcal{L}_u g. \quad (3.24)$$

A vector field v is referred to as a Killing vector field if $\mathcal{K}v = 0$ (i.e., the flow it generates preserves the metric).

3.2.2 Hydrodynamics of active nematic films

In this section, we will extend the mathematical framework from the previous sections to describe the hydrodynamics of an active nematic film on a curved surface. The system can be analyzed in two parts. First, we have the steady-state Stokes equations, which describe how the fluid responds to the active stress induced by the nematic configuration. Subsequently, the steady-state fluid velocity acts as the driving force for the nematodynamics equation, an advection-diffusion equation that governs the

behavior of the nematic field. The system assumes quasi-static coupling, meaning that the dissipation of the fluid at this length scale is considered instantaneous.

Forced surface Stokes flow

We focus on an active nematic confined to a fluid interface, where the flow is governed by the 2D overdamped Stokes equations. Bulk fluid effects are not explicitly modeled, which is valid when surface dissipation dominates over bulk dissipation (i.e., when the Saffman-Delbrück length far exceeds the system size) [6, 100]. The dynamics of such dissipation-driven systems can be systematically derived using the Onsager variational principle [31, 33, 7, 117, 70, 6]. The associated functional, known as the Rayleighian, captures the dissipation rate of the system, and Stokes flow emerges as the stationary condition of this dissipation function. In this work, we consider only viscous dissipation, neglecting reactive stresses from nematic elasticity in advection (cf. Sec. 3.2.2 and [15]). The fluid flow is driven by the active nematic stress $\sigma = \alpha Q \in \Gamma(TM \otimes_{\text{symm}} TM)$, which aligns with the orientation of the nematic constituents and can be either contractile ($\alpha > 0$) or extensile ($\alpha < 0$) [103]. The assumption of an active stress-dominant regime is valid in many practical cases [34, 113].

The formulation of Stokes flow involves minimizing the Rayleighian while satisfying the incompressibility constraint,

$$\min_{\mathbf{u}} \quad \mu \langle \mathcal{K} \mathbf{u}, \mathcal{K} \mathbf{u} \rangle - \langle \mathbf{u}, \text{div}^\nabla \sigma \rangle \quad \text{subject to} \quad \text{div} \mathbf{u} = 0, \quad (3.25)$$

where the first term $\mu \langle \mathcal{K} \mathbf{u}, \mathcal{K} \mathbf{u} \rangle$ quantifies the rate of viscous dissipation [20, 106, 88]. Here the divergence operator, $\text{div} := -\text{grad}^* : \Gamma(TM) \rightarrow C^\infty$, is the negated L_2 adjoint of the gradient operator, $\text{grad} := g^{-1}d : C^\infty \rightarrow \Gamma(TM)$.

The negated adjoint of the covariant derivative is referred to as the covariant divergence, $\text{div}^\nabla := -\nabla^* : \Gamma(TM \otimes TM) \rightarrow \Gamma(TM)$.

The Killing operator $\mathcal{K}$ can be decomposed using the covariant *del bar* operator $\bar{\partial}^\nabla$ as $\mathcal{K}\mathbf{u} = \bar{\partial}^\nabla \mathbf{u} + \text{div}(\mathbf{u})\mathbf{I}/2$. Here, $\bar{\partial}^\nabla \mathbf{u}$ quantifies the degree of non-conformality or infinitesimal shearing induced by the velocity field. The remaining term, $\text{div}(\mathbf{u})\mathbf{I}/2$, characterizes isotropic fluid dilation. For a detailed discussion of the holomorphic structure, including the definitions of covariant *del* ∂^∇ and *del bar* $\bar{\partial}^\nabla$ operators and their relevance in fluid mechanics, see Sec. 3.2.1. Under incompressibility, we can replace $\mathcal{K}$ with $\bar{\partial}^\nabla$ and represent the constrained optimization problem as a minimax problem:

$$\min_{\mathbf{u}} \max_p \mathcal{R} = \mu \langle \bar{\partial}^\nabla \mathbf{u}, \bar{\partial}^\nabla \mathbf{u} \rangle - \langle \mathbf{u}, \text{div}^\nabla \boldsymbol{\sigma} \rangle - \langle p, \text{div} \mathbf{u} \rangle. \quad (3.26)$$

In this expression, p is the fluid pressure acting as a Lagrange multiplier for incompressibility. On a closed surface with $\partial M = \emptyset$, the stationary conditions with respect to velocity $\mathbf{u}$ and pressure p yield the incompressible, steady Stokes equations on a 2D Riemannian manifold:

$$\begin{aligned} 2\mu \bar{\partial}^{\nabla*} \bar{\partial}^\nabla \mathbf{u} + \text{grad } p - \alpha \text{div}^\nabla \mathbf{Q} &= 0, \\ \text{div } \mathbf{u} &= 0. \end{aligned} \quad (3.27)$$

The curvature contributes to the viscous term as $-2\bar{\partial}^{\nabla*} \bar{\partial}^\nabla \mathbf{u} = (\Delta + K)\mathbf{u}$. This result can be deduced from the known approach based on Riemannian geometry that results in the relation $-2\mathcal{K}^* \mathcal{K} \mathbf{u} = (\Delta + K + \text{grad div})\mathbf{u}$ [43, 20, 106]. Additionally, in App. B, we provide an alternative derivation within the framework of complex manifolds using complex differential forms.

Note that the existence of a Killing vector field indicates symmetries in the geometry, such as the rigid body rotation on a rotationally symmetric surface. In such case, the Stokes equations might lack full rank, leading to non-unique solutions with a kernel represented by the Killing vector field. In our study, we select the least-norm solution as the canonical one to the Stokes equations. This choice is equivalent to

projecting out all modes of Killing fields in the L_2 sense. The treatment is justified by the following. In theory, the L_2 Killing component of the velocity does not contribute to energy dissipation, and as a result, it remains conserved in the presence of any non-zero inertia. This is an instance of Noether's theorem, which states that the presence of continuous spatial symmetry, represented by a Killing vector field, implies the conservation of momentum. Therefore, in such ideal case, the correct solution is the least-norm solution with the addition of the Killing mode in the initial condition. However, in practical situations, particularly when considering an interfacial fluid surface that separates two bulk viscous fluids, the Killing mode will eventually dissipate energy through frictional interactions with the substrate¹. In this case, the system should converge to the least-norm solution over time. In Sec. 3.3.2, we will elaborate on the implementation of the Killing projection.

Nematodynamics

Modeling nematodynamics involves studying the dynamics of nematic liquid crystals, which possess both fluid-like and orientational order properties. In this study, we are particularly interested in the sharply aligned limit of the active nematic field. This sharply aligned phase arises when the microscopic rotational diffusion of the nematic molecules is negligible compared to steric alignment [42]. Under these conditions, the nematodynamics of the macroscopic order parameter becomes identical to the dynamics of individual microscopic nematic filaments. A commonly used approach is to utilize an advection-diffusion equation, where advection accounts for the flow-induced alignment and rotation of the nematic director field, while diffusion represents the smoothing of the director due to random molecular interactions [12]. There are various convective rates and models proposed for describing nematic advection and flow alignment [90, 45, 85]. In this work, we model the advection of nematic molecules

¹Interactions with the bulk fluid are not considered in this study.

based on Jeffery’s microphysical theory [54, 42] and express it using the Lie derivative framework. The numerical scheme for this convective rate is detailed in Sec. 3.3.2 and can be easily adapted to accommodate commonly used rates.

The relaxation or diffusion part of the nematodynamics equation involves the nematic Bochner Laplacian established in Sec. 3.2.1. The constraint $||[q]|| = 1$ imposes the sharply-aligned phase. However, according to the Poincaré-Hopf theorem, zeros of the director field are guaranteed unless the system is on a torus with a vanishing Euler characteristic $\chi = 0$ (cf. Sec. 3.2.1). To address this constraint, we adopt a Landau-de Gennes formulation and introduce a weak constraint through a penalizing energy term $\mathcal{G}([q]) = (|[q]|^2 - 1)^2 / (4\varepsilon^2)$ in addition to the Dirichlet energy $|\nabla^L[q]|^2 / 2$. Here, ε represents the coherence length scale around the defects [89]. Together, these terms are referred to as the Frank-Oseen energy [38, 26, 122, 64, 41], whose gradient flow induces a diffusion. Note that the nematodynamics in the close vicinity of defects is not captured by the present model and requires more detailed analysis [19].

The advection of nematic molecules is captured by a combination of an adapted isometric Lie transportation and the “rotation-only” Lie transportation. Lie transportation, as described in Sec. 3.2.1, models how the nematic director field is passively transformed by the flow generated by a velocity field. The rotation-only Lie transportation factors out the stretching part of the Lie transportation, and thus a nematic director is only rotated while keeping its magnitude.

By decomposing the velocity gradient into symmetric ($E = \mathcal{K}u$, representing strain rate) and skew-symmetric ($W = \nabla u - E$, representing vorticity) components, we can express the nematic Lie derivative (cf. Eqn. (3.8) and Sec. 3.2.1) as $\mathcal{L}_u[q] = 2q\mathcal{L}_u q$, where $\mathcal{L}_u q = \nabla_u q - E q - W q$. The vorticity W and the parallel transport ∇_u contribute to infinitesimal rotational transformations that preserve length, whereas the strain rate E affects the molecules by aligning them with the flow and causing axial stretching or compression.

A rotational Lie derivative, $\mathcal{L}_u^{\text{rot}}$, also known as the Jaumann or corotational derivative, specifically focuses on the rotation effect induced by the flow, (e.g., $\mathcal{L}_u^{\text{rot}} \mathbf{q} = \nabla_u \mathbf{q} - \mathbf{W} \mathbf{q}$ for vector fields $\mathbf{q}$). Like the classical Lie derivatives (Sec. 3.2.1), the rotational Lie derivative $\mathcal{L}_u^{\text{rot}}$ is defined through a pullback operation. The only difference is that this pullback is only the rotation component of the classical pullback operator. Recall from Eqn. (3.7) that if φ is a flow map with deformation gradient $\mathbf{F}$, the pullback of a vector field and respectively a nematic field is given by $\varphi^* \mathbf{q} = \mathbf{F}^{-1} \mathbf{q}$, $\varphi^* [\mathbf{q}] = [\mathbf{F}^{-1} \mathbf{q}]$ and $\varphi^* \mathbf{Q} = \mathbf{F}^{-1} \mathbf{Q} \mathbf{F}^{-\top}$. Now, let $\mathbf{R}$ denote the rotational component of $\mathbf{F}$ in the polar decomposition $\mathbf{F} = \mathbf{R} \mathbf{U}$. Define the *rotational pullback* φ_{rot}^* on vector fields and nematic fields by $\varphi_{\text{rot}}^* \mathbf{q} := \mathbf{R}^{-1} \mathbf{q}$, $\varphi_{\text{rot}}^* [\mathbf{q}] := [\mathbf{R}^{-1} \mathbf{q}]$, and $\varphi_{\text{rot}}^* \mathbf{Q} := \mathbf{R}^{-1} \mathbf{Q} \mathbf{R}^{-\top}$. Via this rotational pullback operator, define the Jaumann derivative as $\mathcal{L}_u^{\text{rot}} := \frac{\partial}{\partial t} \circ \varphi_{\text{rot}}^*$, where φ is the infinitesimal flow map generated by $\mathbf{u}$. In terms of the gradients $\nabla \mathbf{u}$, $\mathbf{E}$, $\mathbf{W}$ of $\mathbf{u}$,

$$\mathcal{L}_u^{\text{rot}} \mathbf{q} = \nabla_u \mathbf{q} - \mathbf{W} \mathbf{q}, \quad \mathcal{L}_u^{\text{rot}} [\mathbf{q}] = 2\mathbf{q} \mathcal{L}_u^{\text{rot}} \mathbf{q}, \quad \mathcal{L}_u^{\text{rot}} \mathbf{Q} = \nabla_u \mathbf{Q} - \mathbf{W} \mathbf{Q} + \mathbf{Q} \mathbf{W}. \quad (3.28)$$

The combination of the Jaumann derivative with the traditional Lie derivative in a weighted manner, denoted as $\mathcal{L}_u^\lambda := \lambda \mathcal{L}_u + (1 - \lambda) \mathcal{L}_u^{\text{rot}}$, $\lambda \in [0, 1]$, allows for the modeling of nematic particles with different aspect ratios. The parameter $\lambda \in [0, 1]$ allows for a linear transition from considering the full velocity gradient at $\lambda = 1$ to only the spin tensor at $\lambda = 0$. In the literature, the non-dimensional parameter λ is referred to as the tumbling parameter or Bretherton's constant and determines the dominance of alignment in extensional flow versus tumbling in rotational flow [18, 114].

In addition, nematic molecules are inextensible when subjected to flow. During advection, the magnitude of the order parameter is transported like a scalar field without undergoing stretching. Therefore, we project out the stretching component in the Lie

derivatives $\mathcal{L}_u^\lambda$. Explicitly, the projected Lie derivatives are given by

$$\begin{aligned}
\mathcal{P}(\mathcal{L}_u^\lambda \mathbf{q}) &:= \mathcal{L}_u^\lambda \mathbf{q} - \frac{1}{|\mathbf{q}|^2} \langle \mathcal{L}_u^\lambda \mathbf{q}, \mathbf{q} \rangle \mathbf{q} = \nabla_u \mathbf{q} - \mathbf{W} \mathbf{q} - \lambda \mathbf{E} \mathbf{q} + \frac{\lambda}{|\mathbf{q}|^2} \langle \mathbf{E} \mathbf{q}, \mathbf{q} \rangle \mathbf{q}, \\
\mathcal{P}^L(\mathcal{L}_u^\lambda [\mathbf{q}]) &:= \mathcal{L}_u^\lambda [\mathbf{q}] - \frac{1}{|[\mathbf{q}]|^2} \langle \mathcal{L}_u^\lambda [\mathbf{q}], [\mathbf{q}] \rangle [\mathbf{q}] = 2\mathbf{q} \mathcal{P}(\mathcal{L}_u^\lambda \mathbf{q}), \\
\mathcal{P}^E(\mathcal{L}_u^\lambda \mathbf{Q}) &:= \mathcal{L}_u^\lambda \mathbf{Q} - \frac{1}{|\mathbf{Q}|^2} \langle \mathcal{L}_u^\lambda \mathbf{Q}, \mathbf{Q} \rangle \mathbf{Q} \\
&= \nabla_u \mathbf{Q} - (\lambda \mathbf{E} + \mathbf{W}) \mathbf{Q} - \mathbf{Q}(\lambda \mathbf{E} - \mathbf{W}) + \frac{2\lambda}{|\mathbf{Q}|} \langle \mathbf{E}, \mathbf{Q} \rangle \mathbf{Q}.
\end{aligned} \tag{3.29}$$

Here, we normalize $\mathbf{Q}$ based on the Frobenius matrix product $\langle \cdot, \cdot \rangle$. It is equally valid to use the nematic norm $\langle \cdot, \cdot \rangle_{\mathcal{Q}}$ introduced in Eqn. (3.15). Projections based on both norms are equivalent because $\det(\mathbf{Q}) = 0$, meaning $|\mathbf{Q}| = |\mathbf{Q}|_{\mathcal{Q}}$, and $2 \det(\mathbf{Q}, \mathbf{Q}') = \det(\mathbf{Q}, \mathbf{Q})' = \det(\mathbf{Q})' = 0$, indicating $\langle \mathbf{Q}', \mathbf{Q} \rangle = \langle \mathbf{Q}', \mathbf{Q} \rangle_{\mathcal{Q}}$. In the literature, the generalized isometric advection equation based on Eqn. (3.29) is known as Jeffery's equation [54].

By combining Jeffery advection with the Ginzburg-Landau diffusion, we obtain the nematodynamics equations governing the evolution of the surface nematic:

$$\begin{aligned}
[\dot{\mathbf{q}}] + \mathcal{P}^L(\mathcal{L}_u^\lambda [\mathbf{q}]) &= \frac{1}{\eta} \left(\Delta^L [\mathbf{q}] - \frac{\delta \mathcal{G}}{\delta [\mathbf{q}]} \right), \\
\dot{\mathbf{Q}} + \mathcal{P}^E(\mathcal{L}_u^\lambda \mathbf{Q}) &= \frac{1}{\eta} \left(\Delta^E \mathbf{Q} - d\nu \left[\left[\frac{\delta \mathcal{G}}{\delta [\mathbf{q}]} \right] \right] \right),
\end{aligned} \tag{3.30}$$

where

$$\frac{\delta \mathcal{G}([\mathbf{q}])}{\delta [\mathbf{q}]} = \frac{1}{\varepsilon^2} (|[\mathbf{q}]|^2 - 1) [\mathbf{q}], \tag{3.31}$$

and η is the so-called rotational viscosity (distinct from the microscopic rotational diffusion). Although not explicitly apparent, the Gaussian curvature K plays an implicit role in the Bochner Laplacian through the Levi-Civita connection [43, 60].

System of equations

We summarize and non-dimensionalize the governing equations for the hydrodynamics of an active nematic fluid film. Using the domain length scale, denoted as r , as the characteristic length scale and the time scale $\tau = \mu|\alpha|^{-1}$, we can scale all variables and differential operators. This leads to the dimensionless system given by:

$$(\Delta + K)\mathbf{u} - \text{grad } p \pm \text{div}^\nabla \mathbf{Q} = 0, \quad \text{div } \mathbf{u} = 0, \quad (3.32a)$$

$$[\dot{\mathbf{q}}] + \mathcal{P}^L \mathcal{L}_u^\lambda[\mathbf{q}] = \frac{1}{Pe} \left(\Delta^L[\mathbf{q}] - \frac{1}{\epsilon^2} (|\mathbf{q}|^2 - 1)[\mathbf{q}] \right), \quad (3.32b)$$

where $\lambda \in [0, 1]$ is the tumbling parameter; $Pe = |\alpha|\eta r^2 \mu^{-1}$ is the active Peclet number, which measures the influence of activity-driven Jeffery advection compared to nematic relaxation; $\epsilon = \varepsilon/r$ characterizes the size of a defect core by comparing the coherence length to the system size. Throughout the rest of the study, we solve the non-dimensional version of the system. In the numerical results presented below, we assume the limit $\epsilon \rightarrow 0$ and adopt a normalization procedure to account for the Ginzburg-Landau term.

3.3 Discretization and algorithm

In this section, we will explain the construction of a discrete system for modeling an active fluid on a 2D triangular manifold mesh. We solve Eqn. (3.32) by time splitting (cf. Alg. 3). At each time step, the active force is calculated by taking the covariant divergence of the stress tensor induced by the nematic field. This active force gives rise to a fluid velocity field, which is obtained by solving the discrete Stokes equations (Eqn. (3.32a)) through Augmented-Lagrangian (AL) iteration. Next, the nematodynamics equation (Eqn. (3.32b)) evolves the nematic field with the advection by the Stokes flow velocity field, as well as the relaxation. The advection and relaxation terms

are separated in a splitting procedure. We first compute the evolution of the nematic field Lie-advected by the velocity using an explicit semi-Lagrangian (sL) scheme. The one-step relaxation, based on the Bochner Laplacian operator, is solved using the implicit Euler method to ensure unconditional stability. The projection that removes the stretching component of the Lie derivative and the constraint imposed by the Ginzburg-Landau term are consolidated into a single normalization step. For completeness, we also include a discrete counterpart of the theory of Sec. 3.2.1 about curvature and defects, enabling us to determine the locations and charges of the topological defects. The discretization presented in this work offers several distinctive features:

- The calculations are carried out on a 2D triangular mesh, rather than a volumetric grid, allowing for efficient and accurate representation of the system.
- Tensor fields and differential operators are intrinsically represented using the discrete complex line bundle. This representation, in contrast to the common approach of applying extra tangentiality constraints on Euclidean tensors and operators, allows better accuracy and lower computational cost.
- A generalized semi-Lagrangian method is employed to handle the Lie derivative for both vector and tensor quantities, thereby providing a general procedure for the advection of k -atic directors.
- We utilize isomorphic representations of the nematic field through the Veronese map. This allows us to adapt to the more convenient representation at each stage of the algorithm.

In addition to the theoretical and algorithmic development, we provide an implementation using SideFX Houdini with the VEX language and Python module (see Suppl. C for a brief introduction to the software). Our implementation is accessible through a public repository on Github: <https://is.gd/5Vpflf>.

Algorithm 3. Hydrodynamics of active nematics

```
1: Inputs:  
   Surface  $M$ , curvature  $K$ , time step  $\Delta t$   
2: Initialize:  
   Nematic field  $\mathbf{Q} = \mathcal{V}([q])$   
3: for  $t = 0$  to  $T$  do  
4:   STOKES FLOW( $\mathbf{Q}$ )  
5:    $\mathbf{f} \leftarrow \text{div}^\nabla \mathbf{Q}$   
6:    $\mathbf{u}, p \leftarrow \text{SOLVE}((\Delta + K)\mathbf{u} = \text{grad } p - \mathbf{f}, \text{div } \mathbf{u} = 0)$  ▷ AL iteration  
   (Sec. 3.3.6)  
7:   NEMATODYNAMICS( $\mathbf{u}, \mathbf{Q}$ )  
8:    $\mathbf{Q} \leftarrow \text{SOLVE}(\dot{\mathbf{Q}} + \mathcal{P}^E \mathcal{L}_u^\lambda \mathbf{Q} = 0)$  ▷ Explicit sL (Sec. 3.3.2)  
9:    $[q] \leftarrow \mathcal{V}^{-1}(\mathbf{Q})$   
10:   $[q] \leftarrow \text{SOLVE}(\dot{[q]} = Pe^{-1} \Delta^L [q])$  ▷ Implicit Euler  
11:   $\mathbf{Q} \leftarrow \mathcal{V}(\text{NORMALIZE}([q]))$   
12:   $t \leftarrow t + \Delta t$ 
```

3.3.1 Discrete manifold and choice of variable space

In this section, our goal is to establish the discrete Levi-Civita connection (Sec. 3.3.1) on a discrete triangular mesh and specify the choice of the space of discretization variables (Sec. 3.3.1) for the active nematic system. The notation for mesh navigation is standard in the literature [28] and likely to be clear from the context. To aid the reader, we have also included self-contained introductory materials in Suppl. A for an audience unfamiliar with the notation.

Levi-Civita connection

We equip the discrete manifold with a Levi-Civita connection (cf. Sec. 3.2.1) by locally constructing the discrete parallel transport. This includes the parallel transport from the tangent space of one vertex to its neighboring vertex, from a face to its neighboring face, as well as from a vertex to its neighboring face. The construction of a discrete Levi-Civita connection follows common approaches in discrete differential geometry and design of k -RoSy fields [101, 60, 28].

The face-face and vertex-vertex transportations can be defined intrinsically

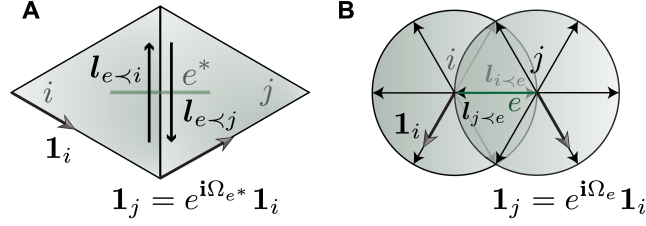


Figure 3.1. Schematics depicting intrinsic parallel transport on a 2D triangular mesh: A) face-face transportation by flattening the dihedral angle within a two-triangle stencil. B) vertex-vertex transportation preserves angles along geodesic paths within the geodesic polar map.

(i.e., invariant under isometric deformation). Therefore, parallel transport of a vector $\mathbf{u}$ from face f_i to face f_j along the dual edge e^* , where $e \prec f_i, f_j$, can be conceptualized as if the dihedral angle at e were flattened (cf. Fig. 3.1A). In this context, parallel transport simplifies to a basis change across face f_i to face f_j ,

$$\Pi_{i \rightsquigarrow j} : T_{f_i} M \rightarrow T_{f_j} M, \quad \Pi_{i \rightsquigarrow j}(\mathbf{u}_i) = \Pi_{i \rightsquigarrow j}(\hat{\mathbf{u}}_i \mathbf{1}_i) = e^{-i\Omega_{e^*}} \hat{\mathbf{u}}_i \mathbf{1}_j. \quad (3.33)$$

The vertex-vertex parallel transport is intrinsically defined by maintaining the angle with respect to the connecting edge e between neighboring vertices $v_i, v_j \prec e$, illustrated by Fig. 3.1B. As seen in Sec. 3.2.1, the nematic field requires amplifying the Levi-Civita connection on the tangent bundle Ω to 2Ω . Subsequently, we can express the parallel transport of a nematic director $[\mathbf{q}]_i \in L_{v_i}$ to L_{v_j} along $\mathbf{l}_e$ as

$$\Pi_{i \rightsquigarrow j} : L_{v_i} \rightarrow L_{v_j}, \quad \Pi_{i \rightsquigarrow j}([\mathbf{q}]_i) = \Pi_{i \rightsquigarrow j}([\hat{\mathbf{q}}]_i [\mathbf{1}]_i) = e^{-i2\Omega_e} [\hat{\mathbf{q}}]_i [\mathbf{1}]_j. \quad (3.34)$$

Here, the primal edge connection, $\Omega_e = \arg(\mathbf{l}_{v_j \prec e}) - \arg(\mathbf{l}_{v_i \prec e}) - \pi$, quantifies the basis change across $T_{v_i} M$ and $T_{v_j} M$ (cf. Fig. 3.1B).

The vertex-face transportation is achieved using the $\mathbb{R}^3$ embedding. Given the embedding of the mesh and the realization of the tangent vector in $\mathbb{R}^3$, we can utilize a 3D dihedral rotation to map from one vertex tangent space to a neighboring face

tangent space and vice versa (cf. the continuous theory in Sec. 3.2.1). To illustrate this, consider a vertex unit normal, $\mathbf{n}_v \in T_v^\perp M, |\mathbf{n}_v| = 1$, and a face unit normal, $\mathbf{n}_f \in T_f^\perp M, |\mathbf{n}_f| = 1$. The dihedral rotation from $\mathbf{n}_v$ to $\mathbf{n}_f$ is defined as the rotation R_{vf} of with angle $\cos^{-1}\langle \mathbf{n}_v, \mathbf{n}_f \rangle$ about the axis $\mathbf{l} = \mathbf{n}_v \times \mathbf{n}_f / |\mathbf{n}_v \times \mathbf{n}_f|$. Define $\Pi_{v \rightsquigarrow f} := R_{vf} : T_v M \rightarrow T_f M$. Note that the definition of $\mathbf{n}_v$ is not unique. In practice, we use an angle-weighted vertex normal $\mathbf{n}_v := (\sum_{f \succ v} \tilde{\angle}_{f \succ v} \mathbf{n}_f) / (2\pi)$.

Space of system variables and operator mappings

In this section, we specify the choice of space where the discretization variable resides, and the mapping spaces imposed by the differential operators. In Secs. 3.3.2-3.3.4, we will recapitulate the specific definitions of the operators.

In our approach, we discretize the Stokes equations on the faces of the mesh and the nematodynamics equation on the vertices. Specifically, we work with a face-based fluid velocity denoted as $\mathbf{u} \in \Gamma_{\mathfrak{F}}(TM)$ and a vertex-based nematic director represented as $[\mathbf{q}] \in \Gamma_{\mathfrak{V}}(L)$, alongside the corresponding matrix representation $\mathbf{Q} = \mathbf{q} \otimes \mathbf{q} \in \Gamma_{\mathfrak{V}}(E)$.

Subsequently, we develop the discrete differential operators associated with these choices. For the Stokes equations, we construct the Bochner Laplacian $\Delta : \Gamma_{\mathfrak{F}}(TM) \rightarrow \Gamma_{\mathfrak{F}}(TM)$ based on the face-face Levi-Civita connection. The gradient operator, $\text{grad} : C_{\mathfrak{V}}(M) \rightarrow \Gamma_{\mathfrak{F}}(TM)$, maps scalar vertex data to vector face data. The divergence operator, the adjoint of the gradient, maps vectors from faces to scalar measurements at vertices, $\text{div} : \Gamma_{\mathfrak{F}}(TM) \rightarrow C_{\mathfrak{V}}(M)$. Similarly, for the nematodynamics equation, we establish the nematic Bochner Laplacian $\Delta^L[\mathbf{q}] : \Gamma_{\mathfrak{V}}(L) \rightarrow \Gamma_{\mathfrak{V}}(L)$ using the vertex-vertex Levi-Civita connection. We also introduce a vertex-based Lie derivative $\mathcal{L}_{\mathbf{u}} : \Gamma_{\mathfrak{V}}(E) \rightarrow \Gamma_{\mathfrak{V}}(E)$ to advect the nematic director.

3.3.2 Semi-Lagrangian Lie advection and nematodynamics

The sL method is an explicit space-time integrator for advection equations similar to an upwinding scheme [52]. Here we introduce the sL method generalized to any Lie advection equation for tensor fields as follows. Given a velocity field, we construct a backward flowmap by tracing the velocity field upwind. This flowmap is used to pullback the tensor field from the previous time step as the update rule. The principle can be applied to tensors of arbitrary (r, s) type (e.g., the valence- k tensor that represents the k -atic field [45]), serving as a discrete counterpart to the continuous theory discussed in Sec. 3.2.1. The resultant scheme, summarized in Alg. 4, captures the geometric structure of nematic advection and enhances numerical stability.

Concretely, given a time-independent velocity field $\mathbf{u} \in \Gamma_{\mathfrak{W}}(TM)$, we construct a finite-time backward flowmap $\Psi_{\Delta t}$ by integrating the ODE, $\partial_{\tau}\Psi_{\tau} = -\mathbf{u}(\Psi_{\tau})$, $\tau \in [0, \Delta t]$, with the initial condition $\Psi_0 = \text{id}_M$. In practice, we represent Ψ_{τ} and $\mathbf{u}$ as $\mathbb{R}^3$ -valued functions using surface embeddings and implement the integration using RK4. To evaluate $\mathbf{u} \circ \Psi_{\tau}$, we use a closest-point projection followed by a linear interpolation in a triangle. Recall in the continuous setting that Lie advection $\partial_t[\mathbf{q}] + \mathcal{L}_{\mathbf{u}}[\mathbf{q}] = 0$ is characterized by a constant pullback field $\partial_t\varphi^*[\mathbf{q}] = 0$ (cf. Eqn. (3.8)). With the backward flowmap $\Psi_{\Delta t}$ and its gradient $d\Psi_{\Delta t}$, a discrete Lie advection is analogously represented by updating the nematic field $[\mathbf{q}]$ using its pullback value $\Psi_{\Delta t}^*[\mathbf{q}]$. The procedure will be made explicit in following subsections.

Deformation gradient of backward flowmap

By evaluating the finite-time flowmap over vertices (i.e., $\Psi_{\Delta t}$ such that $M_{\Delta t} = \Psi_{\Delta t}M$), we can construct the deformation gradient $d\Psi_{\Delta t}|_f$ (realized in $\mathbb{R}^3$) on each

triangle face f as follows:

$$d\Psi_{\Delta t} : \Gamma_{\mathfrak{F}}(TM) \rightarrow \Gamma_{\mathfrak{F}}(T_{\Psi_{\Delta t}}M_{\Delta t}), \quad d\Psi_{\Delta t}|_f := \begin{bmatrix} l_1 & l_2 & n \\ | & | & | \\ | & | & | \end{bmatrix}_{\Psi_{\Delta t}(f)} \begin{bmatrix} l_1 & l_2 & n \\ | & | & | \\ | & | & | \end{bmatrix}_f^{-1}. \quad (3.35)$$

These column vectors, illustrated by Fig. 3.2, represent the embedded edge vectors $l \in \mathbb{R}^3$ and the unit face normal $n \in \mathbb{R}^3$ before and after the backward flowmap $\Psi_{\Delta t}$. A vertex-based deformation gradient $F : \Gamma_{\mathfrak{V}}(TM) \rightarrow \Gamma_{\mathfrak{V}}(T_{\Psi_{\Delta t}}M_{\Delta t})$ is obtained by averaging the deformation gradient on incident faces, $F|_v = \frac{1}{|f \succ v|} \sum_{f \succ v} d\Psi_{\Delta t}|_f$.

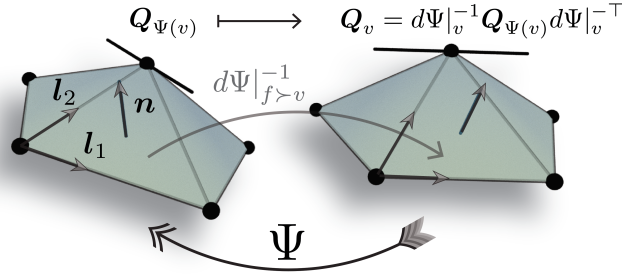


Figure 3.2. The sL Lie advection is performed on a triangular mesh using a backward finite time flowmap Ψ . This flowmap engenders a piecewise constant deformation gradient $d\Psi|_{f \succ v}$ defined over the triangle faces. By averaging this face-based deformation gradient around the one-ring neighborhood of each vertex, we derive a vertex-based deformation gradient $d\Psi|_v$. The nematic Lie advection is modeled by updating Q_v with $d\Psi|_v^{-1} Q_{\Psi(v)} d\Psi|_v^{-T}$.

Pullback of the nematic field

Recall that the nematic field can be equivalently expressed using the matrix representation $Q = \mathcal{V}([q]) \in \Gamma_{\mathfrak{V}}(TM \otimes TM)$. With Q realized as an $\mathbb{R}^{3 \times 3}$ -valued function, its Lie advection, $\partial_t Q + \mathcal{L}_u Q = 0$, with $t \in [0, \Delta t]$, $Q|_{t=0} = Q_0$, can be modeled by

$$Q_{\Delta t} = \Psi_{\Delta t}^* Q_0 = F^{-1}(Q_0 \circ \Psi_{\Delta t}) F^{-T}, \quad (3.36)$$

where $Q_{\Delta t}$ denotes the updated value of Q for the subsequent time step. Because the backward flowmap $\Psi_{\Delta t}$ can land anywhere on M , evaluating $Q_0 \circ \Psi_{\Delta t}$ requires interpolation, such as trilinear interpolation utilizing the face-barycentric representation of $\Psi_{\Delta t}$. However, this interpolation may not preserve the rank-1 structure of Q . To address this, we perform a singular value decomposition (SVD) and extract the leading component of the interpolated value, ensuring that $Q_0 \circ \Psi_{\Delta t} = q_{\Psi_{\Delta t}} \otimes q_{\Psi_{\Delta t}}$, and thereby $Q_{\Delta t}$ remains of rank 1. Lastly, by projecting $Q_{\Delta t}$ onto the surface using tangent bases, we recover the intrinsic representation of $Q_{\Delta t}$ and its corresponding complex representation $[q]_{\Delta t} = \mathcal{V}^{-1}(Q_{\Delta t})$.

To model the Jeffery equation $\dot{Q} + \mathcal{P}^E(\mathcal{L}_u^\lambda Q) = 0$ (cf. Eqn. (3.29)), we can include the additional rotational Lie advection $\mathcal{L}_u^{\text{rot}}$ and normalization as follows:

$$Q_{\Delta t} = \frac{\lambda F^{-1}(Q_0 \circ \Psi_{\Delta t})F^{-\top} + (1 - \lambda)R^{-1}(Q_0 \circ \Psi_{\Delta t})R^{-\top}}{|\lambda F^{-1}(Q_0 \circ \Psi_{\Delta t})F^{-\top} + (1 - \lambda)R^{-1}(Q_0 \circ \Psi_{\Delta t})R^{-\top}|}, \quad (3.37)$$

where R isolates rotation in F through polar decomposition $F = RU$. By omitting the normalization step and adjusting the value of λ , we can adapt the Jeffery advection to other commonly used convective rates, including the standard Lie derivative ($\lambda = 1$) and the Jaumann corotational derivative ($\lambda = 0$).

As a side remark, in the special case where the transported field h is a scalar function, its advection $\partial_t h + \mathcal{L}_u h = \partial_t h + \nabla_u h = 0$ simplifies to the traditional sL method, $h_{\Delta t} = \Psi_{\Delta t}^* h_0 = h_0 \circ \Psi_{\Delta t}$.

Discussion of sL advection scheme

Although the traditional scalar sL method guarantees unconditional stability, the extension to tensor Lie advection in general does not [74]. In these cases, stability depends on the pullback map's conditioning, requiring a CFL-like condition to be met. However, by incorporating a diffusion term into the sL advection through time-

Algorithm 4. sL method for nematic Lie advection over time interval $[0, \Delta t]$

```

1: Inputs:
   Surface mesh  $M$ , nematic field  $\mathbf{Q}_0 = \mathcal{V}([\mathbf{q}]_0)$ , time step  $\Delta t$ , flow
   velocity  $\mathbf{u}$ 
2: for each vertex  $v \in \mathfrak{V}$ , do
3:    $\Psi_{\Delta t} \leftarrow \text{RK4}(d\Psi_\tau/d\tau = -\mathbf{u}(\Psi_\tau))$  ▷ Backward flowmap
4:    $\mathbf{Q}_{\Psi_{\Delta t}} \leftarrow \text{INTERP}(\mathbf{Q}_0 \circ \Psi_{\Delta t})$ 
5:    $\mathbf{Q}_{\Psi_{\Delta t}} \leftarrow \text{SVD}(\mathbf{Q}_{\Psi_{\Delta t}})$ 
6: for each face  $f \in \mathfrak{F}$  do ▷ Deformation gradient
7:    $d\Psi_{\Delta t} \leftarrow [l_1, l_2, n]_{\Psi_{\Delta t}} [l_1, l_2, n]^{-1}$ 
8: for each vertex  $v \in \mathfrak{V}$ , do ▷ Pullback
9:    $\mathbf{F} \leftarrow \text{AVG}(d\Psi_{\Delta t}|_{f \succ v})$ 
10:   $\mathbf{Q}_v \leftarrow \mathbf{F}^{-1} \mathbf{Q}_{\Psi_{\Delta t}} \mathbf{F}^{-\top}$ 

```

splitting (cf. Alg. 3) and scaling the system with respect to advection, the time step for advection remains fixed regardless of the Péclet number. Because the diffusion equation, $[\dot{\mathbf{q}}] = Pe^{-1} \Delta^L [\mathbf{q}]$, is solved using an implicit method (cf. Alg. 3), the algorithm is numerically stable with high Pe . However, as Pe increases, maintaining accuracy requires resolving smaller physical length scales (which scale with $\sqrt{1/Pe}$), which in turn necessitates smaller time steps to satisfy the CFL condition.

We also acknowledge some known limitations inherent to time-splitting and sL method. First, time-splitting introduces errors, which can be reduced using techniques such as the Strang method [52]. Second, the sL method based on linear interpolation has significant numerical diffusion. Improvements can be made by implementing higher-order interpolation and techniques such as the McCormack scheme [52]. However, these challenges are common to the sL method and are not unique to the generalization to tensor-valued functions. Despite these challenges, the current minimal implementation is sufficient for our purpose of studying active nematics on curved surfaces.

3.3.3 Gradient and covariant divergence

We used the linear finite element hat function Φ_v to discretize the gradient for a scalar function $p \in C_{\mathfrak{V}}(M)$ as follows:

$$\text{grad} : C_{\mathfrak{V}}(M) \rightarrow \Gamma_{\mathfrak{F}}(TM), \quad A_f(\text{grad } p)_f = A_f \sum_{v \prec f} p_v (\text{grad}_f \Phi_v) = \frac{1}{2} \sum_{v \prec f} p_v \mathbf{i}l_{v \prec f}, \quad (3.38)$$

where $\mathbf{i}l_{v \prec f}$ denotes the 90° rotation of edge vector $l_{v \prec f}$ and A_f is the face area (cf. Suppl. A for mesh notation details).

The divergence operator is given by the negated adjoint operation of the gradient. The adjoint of Eqn. (3.38) applied to a face-based vector field $\mathbf{u} \in \Gamma_{\mathfrak{F}}(TM)$ is a measure on each vertex $(\text{grad}^* \mathbf{u})_v = -\frac{1}{2} \sum_{f \succ v} \langle \mathbf{i}l_{v \prec f}, \mathbf{u}_f \rangle$, representing the total divergence at vertex v . Using a reference vertex area measure A_v we define the divergence operator as

$$\text{div} : \Gamma_{\mathfrak{F}}(TM) \rightarrow C_{\mathfrak{V}}(M), \quad A_v(\text{div } \mathbf{u})_v = -\frac{1}{2} \sum_{f \succ v} \langle \mathbf{i}l_{v \prec f}, \mathbf{u}_f \rangle. \quad (3.39)$$

Note that the discrete equation involving the divergence operator is often written as in the weak form, where divergence is treated as a measure. In particular these discrete equations are independent of the specific choice of the vertex area measure A_v . In cases where pointwise evaluation is needed, such as in visualization, we use $A_v = \sum_{f \succ v} A_f / 3$.

The divergence operator, represented by Eqn. (3.39), can also be understood as a finite-volume discretization applied to the dual cells of the mesh [52]. We can analogously obtain a vertex-based divergence $\text{div} : \Gamma_{\mathfrak{V}}(TM) \rightarrow C_{\mathfrak{F}}(M)$ by applying finite volume on the primal mesh.

By incorporating an additional vertex-face parallel transport, $\Pi_{v \rightsquigarrow f} : T_v M \rightarrow T_f M$, (cf. Sec. 3.3.1), we can extend such construction to discretize the covariant diver-

gence:

$$\operatorname{div}^\nabla : \Gamma_{\mathfrak{F}}(TM \otimes TM) \rightarrow \Gamma_{\mathfrak{F}}(TM), \quad A_f(\operatorname{div}^\nabla \mathbf{Q})_f = -\frac{1}{2} \sum_{v \prec_f} \Pi_{v \rightsquigarrow f} \left[\mathbf{Q}_v \cdot \Pi_{v \rightsquigarrow f}^{-1}(\mathbf{i}l_{v \prec f}) \right]. \quad (3.40)$$

The Euclidean dot product, denoted as $\cdot$, denotes a contraction operation on one slot of the $(2,0)$ tensor (i.e., $\mathbf{Q} \cdot \mathbf{q} \equiv Q^{ij}q_j$). In the case of $\mathbf{Q} = \mathbf{q} \otimes \mathbf{q}$ being rank-1, the covariant divergence can be explicitly expressed, $A_f(\operatorname{div}^\nabla \mathbf{Q})_f = -\frac{1}{2} \sum_{v \prec_f} \langle \Pi_{v \rightsquigarrow f}(\mathbf{q}_v), \mathbf{i}l_{v \prec f} \rangle \Pi_{v \rightsquigarrow f}(\mathbf{q}_v)$. In practice, these discrete operators, including those in Sec. 3.3.4, are organized and computed using sparse matrix multiplication.

3.3.4 Bochner Laplacian

Using the face-face parallel transport $\Pi_{i \rightsquigarrow j}$, $i, j \in \mathfrak{F}$ across edge $e \prec i, j$ (cf. Eqn. (3.33)), the discrete directional covariant derivative ∇_{l^*} maps a face-based vector field $\mathbf{u} = \hat{\mathbf{u}}\mathbf{1} \in \Gamma_{\mathfrak{F}}(TM)$ to an edge-based vector field,

$$\nabla_{l^*} : \Gamma_{\mathfrak{F}}(TM) \rightarrow \Gamma_{\mathfrak{E}}(TM), \quad |l_e^*|(\nabla_{l^*}\mathbf{u})_e = (\mathbf{u}_j - \Pi_{i \rightsquigarrow j}(\mathbf{u}_i))_{i,j \succ e} = \left((\hat{u}_j - e^{-i\Omega_e^*} \hat{u}_i) \mathbf{1}_j \right)_{i,j \succ e}. \quad (3.41)$$

By taking the L_2 norm with the edge area as the diamond area formed by l and l^* , $A_e = |l||l^*|$, we can construct the discrete Dirichlet energy (cf. Eqn. (3.13)) as:

$$2\mathcal{F}(\mathbf{u}) = \sum_{e \in \mathfrak{E}} A_e |\nabla_{l^*}\mathbf{u}|_e^2 = \sum_{e \in \mathfrak{E}} \frac{A_e}{|l_e^*|^2} |\mathbf{u}_j - \Pi_{i \rightsquigarrow j}(\mathbf{u}_i)|_{i,j \succ e}^2 = \sum_{e \in \mathfrak{E}} \frac{1}{w_e} |\hat{u}_j - e^{-i\Omega_e^*} \hat{u}_i|_{i,j \succ e}^2. \quad (3.42)$$

The edge length ratio $w_e = |l^*|/|l| = (\cot \angle_{e \prec i} + \cot \angle_{e \prec j})/2$ is commonly referred to as the cotangent weight, where $\angle_{e \prec i}$ refers to the corner angle opposite to the edge e at face i .

The discrete Bochner Laplacian can be obtained by taking the negated variation of the discrete Dirichlet energy,

$$\Delta : \Gamma_{\mathfrak{F}}(TM) \rightarrow \Gamma_{\mathfrak{F}}(TM), \quad A_f(\Delta \mathbf{u})_f = - \left(\frac{\delta \mathcal{E}}{\delta \mathbf{u}} \right)_f = - \mathbf{1}_f \sum_{e \prec f} \frac{1}{w_e} (\hat{u}_f - e^{-i\Omega_e} \hat{u}_i)_{i,f \succ e}. \quad (3.43)$$

Here, face i is in the one-ring neighbor of face f , and i and f share the edge e .

Similarly, using the vertex-vertex transportation $\Pi_{i \rightsquigarrow j}$, $i, j \in \mathfrak{V}$ across edge $e \succ i, j$ (cf. Eqn. (3.34)), we can construct the covariant derivative for a vertex-based nematic field as follows:

$$\nabla_l^L : \Gamma_{\mathfrak{V}}(TM) \rightarrow \Gamma_{\mathfrak{E}}(TM), \quad |l_e|(\nabla_l[\mathbf{q}])_e = ([\mathbf{q}]_j - \Pi_{i \rightsquigarrow j}([\mathbf{q}]_i))_{i,j \prec e} = \left((\hat{q}_j - e^{-i2\Omega_e} \hat{q}_i)[\mathbf{1}]_j \right)_{i,j \prec e}. \quad (3.44)$$

The nematic Bochner Laplacian acting on the vertex-based nematic field $[\mathbf{q}] = [\hat{q}][\mathbf{1}] \in \Gamma_{\mathfrak{V}}(L)$ is therefore:

$$\Delta^L : \Gamma_{\mathfrak{V}}(L) \rightarrow \Gamma_{\mathfrak{V}}(L), \quad A_v(\Delta[\mathbf{q}])_v = -[\mathbf{1}]_v \sum_{e \prec v} w_e ([\hat{q}]_v - e^{-i2\Omega_e} [\hat{q}]_i)_{i,f \prec e}, \quad (3.45)$$

where vertex i is in the one-ring neighbor of vertex v , connected by edge e .

Strictly speaking, there is an ambiguity in the definition of the directional covariant derivative (Eqns. (3.41) and (3.44)) because it is equally valid to transport a vector (nematic director) from face (vertex) j to face (vertex) i before applying finite difference. However, the discretization of the Dirichlet energy, and thus the Bochner Laplacian (Eqns. (3.43) and (3.45)), relies only on the norm of the covariant derivative, which remains invariant, $|\Pi_{j \rightsquigarrow i}(\mathbf{u}_j) - \mathbf{u}_i| = |\mathbf{u}_j - \Pi_{i \rightsquigarrow j}(\mathbf{u}_i)|$, $|\Pi_{j \rightsquigarrow i}([\mathbf{q}]_j) - [\mathbf{q}]_i| = |[\mathbf{q}]_j - \Pi_{i \rightsquigarrow j}([\mathbf{q}]_i)|$.

In summary, the intrinsic definition of the discrete Bochner Laplacian emerges naturally from continuous theory of complex line bundles. Contrasting the previous covariant divergence (Eqn. (3.40)) that relies on an extrinsic parallel transport achieved through dihedral rotations between the tangent spaces of vertices and faces, the Bochner Laplacian is constructed purely intrinsically by vertex-vertex or face-face parallel transport. Compared to common approaches that rely on the embedding structure in $\mathbb{R}^3$ and a projection operator from $\mathbb{R}^3$ to the manifold, the discrete complex line bundle approach has notably improved efficiency. This approach reduces the degree of freedom per element from 3 to 2, and the complex nature furthermore takes advantage of the optimized complex arithmetic handling in established numerical linear algebra libraries. The corresponding complex-valued Laplace matrix is Hermitian and negative-semidefinite.

3.3.5 Curvature, defects, and local-global geometry

Here we introduce a discretization of the face Gaussian curvature and an algorithm to detect topological defects and compute their charges. This serves as the discrete counterpart to the continuous theory in Sec. 3.2.1.

Given a vertex-based k -atic field $[q]_k : \mathfrak{V} \rightarrow V_k$, we can utilize vertex-vertex parallel transport $\Pi_{i \rightarrow j}$ (cf. Eqn. (3.34)) with $i, j \in \mathfrak{V}$ along the incident edge $e \succ i, j$ to measure the finite rotation $|l_e|\eta_e$ (cf. Eqn. (3.17)) of $[q]_k$ along e as follows:

$$|l_e|\eta_e = \arg \left(\frac{[q]_{k,j}}{\Pi_{i \rightsquigarrow j}[q]_{k,i}} \right) = \arg \left(\frac{[\hat{q}]_{k,j}}{e^{-ik\Omega_e}[\hat{q}]_{k,i}} \right). \quad (3.46)$$

Assuming the k -atic field $[q]_k$ is continuous between vertices i and j , we choose the principal branch of the $\arg$ function so that $|l_e|\eta_e$ lies within $(-\pi, \pi]$.

Recall that in the continuous setting, Gaussian curvature quantifies holonomy, $\int_U K dA = - \int_{\partial U} \omega$ (cf. Sec. 3.2.1). Analogously, discrete Gaussian curvature $K \in C_{\mathfrak{F}}(M)$

can be represented by closed-loop summation of the primal edge connection Ω_e as

$$A_f K_f := - \sum_{e \prec f} \Omega_e \mod 2\pi \in (-\pi, \pi]. \quad (3.47)$$

Selecting branch $(-\pi, \pi]$ for $A_f K_f$ is justified by the small integration domain of face f . The discrete curvature is also the angle defect at face f , given by $A_f K_f = \sum_{v \prec f} \tilde{Z}_v - \pi$, where $\tilde{Z}_v$ is the geodesic corner angle (cf. Suppl. A). The discrete Gauss-Bonnet theorem states, $\sum_f A_f K_f = 2\pi\chi(M)$, where $\chi(M) = |\mathfrak{V}| - |\mathfrak{E}| + |\mathfrak{F}| = 2 - 2g$ represents the Euler characteristic of the polygonal mesh M .

Analogous to Eqn. (3.18) in the continuous theory, given a k -atic field $[q]_k$, its topological charge is a fraction-valued function $\mathcal{Z}([q]_k) : \mathfrak{F} \rightarrow \mathbb{Z}/k$ that can be computed for each face f as follows:

$$\mathcal{Z}_f([q]_k) = \frac{1}{2\pi k} \sum_{e \prec f} |l_e| \eta_e + \frac{1}{2\pi} A_f K_f. \quad (3.48)$$

The charge $\mathcal{Z}_f$ at face f is zero when the adjacent k -atic directors $[q]_{k,v \prec f}$ are continuous, and nonzero in the presence of defects. The charge function satisfies the discrete Poincaré-Hopf theorem exactly, $\sum_f \mathcal{Z}_f([q]_k) = \chi(M)$.

3.3.6 Augmented Lagrangian method for the Stokes equations

We utilize a classic AL iteration to solve the incompressible Stokes flow [93]. Recall that the Stokes equations can be viewed as the stationary condition for a divergence-constrained variational problem (cf. Sec. 3.2.2). The AL approach augments the original Rayleighian with a quadratic penalty term on the divergence. The augmented Rayleighian is given by:

$$\min_u \max_p \mathcal{R}_{\text{AL}} = \mathcal{R}(\mathbf{u}, p) + \frac{k}{2} \langle \text{div } \mathbf{u}, \text{div } \mathbf{u} \rangle, \quad (3.49)$$

where the stationary condition is $(\Delta + K + k \text{ grad} \circ \text{div})\mathbf{u} - \text{grad } p + \mathbf{f} = 0$. Here, the penalty coefficient k enforces the incompressibility constraint during the iterative solution process. Shown in Alg. 5, the solve process involves iteratively solving the modified Stokes equations. At each iteration, the velocity and pressure fields are updated until a convergence criterion is met. Practically, increasing k makes the fluid very stiff in response to compressibility, which leads to convergence in a small number of iterations. However, a high value of k also increases the spectral condition number of $(\Delta + K + k \text{ grad} \circ \text{div})$, which increases the time per iteration [93].

Algorithm 5. AL subroutine for incompressible Stokes equations

```

1: Inputs:
   Penalty coefficient  $k$ , tolerance  $\varepsilon$ 
2: Initialize:
   Initial guess  $p$ 
3: while  $\|\text{div } \mathbf{u}\| > \varepsilon$  do
4:    $\mathbf{u} \leftarrow \text{SOLVE}(-(\Delta + K + k \text{ grad} \circ \text{div})\mathbf{u} = \mathbf{f} - \text{grad } p) \triangleright$  Poisson solve ( $\mathbf{A} \mathbf{x} = \mathbf{b}$ )
5:    $p \leftarrow p + k \text{ div } \mathbf{u}$ 
6:
7:   if exist  $N$  distinct  $\omega_i$  then  $\triangleright$  Remove the Killing component
8:     for  $i = 1$  to  $N$  do
9:        $\mathbf{u} \leftarrow \mathbf{u} - \langle\langle \mathbf{u}, \omega_i \rangle\rangle \omega_i$ 

```

As discussed in Sec. 3.2.2, when dealing with the Stokes equations in symmetric geometries that possess a Killing vector field, the solution is not unique. To ensure that we obtain continuous solution during time evolution, we consistently select the solution with the minimal L_2 norm by projecting out components that lie in the space of the Killing vector fields. To achieve this, we precompute the set of Killing bases of the geometry, denoted as ω_i , before we begin the evolution of the active nematic system. The bases are determined by solving the nonvanishing vector field that satisfies $(\Delta + K + \text{grad} \circ \text{div})\omega_i = 0$. In practice, we solve this equation computationally through the smallest eigenvalue problem and extract eigenfunctions with near-zero eigenvalues. Here, each Killing basis is normalized such that $\langle\langle \omega_i, \omega_j \rangle\rangle = \delta_{ij}$. As outlined in Alg. 5,

we project out components spanned by these bases after the AL iteration. This protocol is carried out in Sec. 3.4 for examples on the sphere and torus.

3.4 Results and discussion

In the preceding sections, we established the continuous formulation, its corresponding discretization, and the numerical methods employed. In this section, we demonstrate the robustness of the method and its potential in addressing various biophysically relevant scenarios. First, we conduct a convergence analysis spanning the various subroutines of our methodology. This analysis not only attests to the method’s accuracy in approximating the smooth solution but also illustrates the performance of the algorithms. In addition, we explore the efficiency and scalability of the methods by presenting the computational-time scaling characteristics of the major components within the framework. We have chosen to benchmark these methods on simple geometries, where a mix of numerical experiments and analytical comparisons are possible. However, the application extends beyond idealized geometries. To demonstrate the generality of these methods on arbitrary geometries and topologies, we apply them to solve the full nematodynamics equations in biophysically relevant examples. In this section, through the error analysis, scalability metrics, and examples, we expect to demonstrate the reliability and versatility of the method.

3.4.1 Convergence analysis

We perform convergence tests for our method using a combination of analytical results and numerical experiments. We break down the method into different steps, including the Lie advection of the nematic field and the solution of the surface Stokes equations.

We start by testing the spatial convergence of the nematic semi-Lagrangian advection on a unit sphere. To do this, we parameterize the unit sphere using spherical

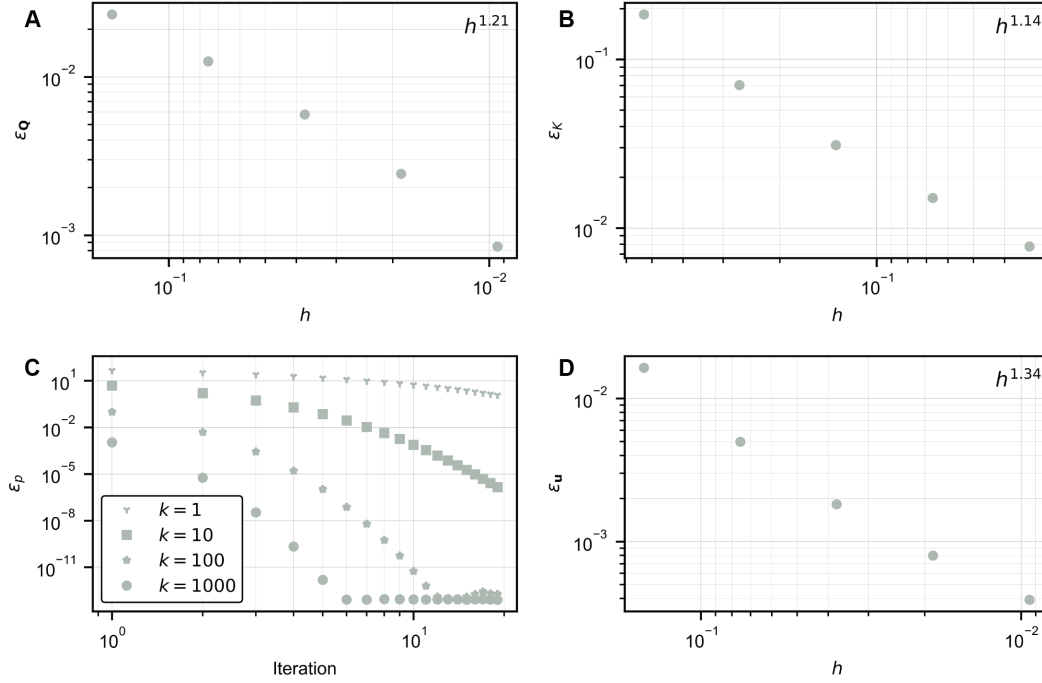


Figure 3.3. Convergence test of different aspects of the method: A) Nematic sL advection error with mesh refinement; B) Gaussian curvature error with mesh refinement; C) Area dilation of the fluid velocity solution after each AL pressure iteration; and D) Error of Stokes equations solution with mesh refinement.

coordinates, with polar angle $\theta \in [0, \pi]$ and azimuthal angle $\varphi \in [0, 2\pi)$, which results in the spherical vector bases, $e_\theta := \partial_\theta / |\partial_\theta|$ and $e_\varphi := \partial_\varphi / |\partial_\varphi|$. We prescribe an analytical nematic field along the unit polar basis vector e_θ , which creates two $+1$ defects at the poles of the sphere. We also prescribe an advecting velocity $\mathbf{u}$, which is defined using solenoidal vector spherical harmonics with $l = 2, m = 0$ (i.e., $\mathbf{u} = \boldsymbol{\phi}_{20} = (3/2)\sqrt{5/\pi} \sin \theta \cos \theta e_\theta$). We then advect the nematic field for one time step using step size $\tau = 0.1$ to measure the local error. We use the numerical result obtained from the finest mesh tested as the ground truth, denoted as $\overline{\mathbf{Q}}$, and compare results from coarser meshes using the L_2 relative error norm, $\varepsilon_Q := \|\mathbf{Q} - \overline{\mathbf{Q}}\| / \|\overline{\mathbf{Q}}\|$. As shown in Fig. 3.3A, the error decreases approximately linearly with mesh refinement, following $\varepsilon_Q \propto h^{1.21}$, where h represents the average edge length of the mesh. Note that here we omitted the unit-length normalization step in Alg. 3 and directly compared the embedded $\mathbb{R}^{3 \times 3}$

matrix representation of $\mathbf{Q}$.

The Stokes momentum equation involves the estimation of Gaussian curvature K . To test this, we use a torus for which the analytical expression of Gaussian curvature is known, with a value that varies from negative to positive. We consider a torus with a radius of revolution R , and a radius of cross-section r , for which the Gaussian curvature can be parametrized using the polar coordinate as $\bar{K}(\theta) = \cos \theta / (rR + r^2 \cos \theta)$. We calculate the face-based Gaussian curvature K using Eqn. (3.47) and compare it to the analytical results using a relative L_2 error, denoted as $\varepsilon_K := \|K - \bar{K}\| / \|\bar{K}\|$, where $\bar{K}(\theta)$ is computed at the center of the face. Fig. 3.3B demonstrates that the error decreases linearly with mesh refinement, following $\varepsilon_Q \propto h^{1.14}$.

We also evaluate the procedure of the AL that updates the pressure p to enforce the incompressibility constraint. This involves measuring the pressure error norm that is defined by the the norm of area dilation, $\varepsilon_p := |\text{div } \mathbf{u}|$. Fig. 3.3C illustrates that increasing the penalty coefficients k results in faster convergence to a divergence-free solution. However, this comes at the cost of making the matrix inversion ill-conditioned, which increases the time per iteration.

Finally, we compare the numerical solution of the Stokes equations with its analytical counterpart. We construct the forcing term $\mathbf{f}$ in the momentum equation using the divergence-free spherical harmonics basis function, which is an eigenfunction of the Stokes equations. This approach allows us to derive an analytical solution $\bar{\mathbf{u}}$ for the surface flow on a sphere. We chose $\mathbf{f}$ to be a mix of two eigenmodes, $\mathbf{f} = \boldsymbol{\phi}_{21} + \boldsymbol{\phi}_{20}$, and generated numerical solutions $\mathbf{u}$ across different mesh resolutions. The error, as illustrated in Fig. 3.3D, decreases linearly with the refinement of the mesh, following the relation $\varepsilon_u := \|\mathbf{u} - \bar{\mathbf{u}}\| / \|\bar{\mathbf{u}}\| \propto h^{1.34}$.

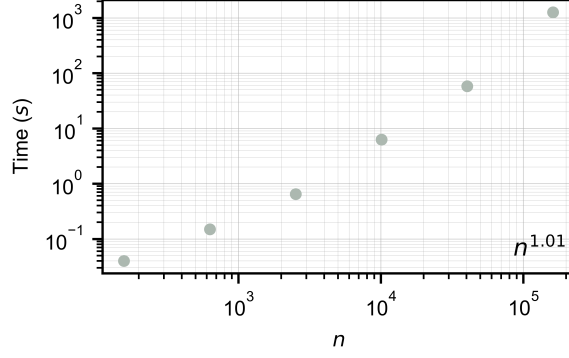


Figure 3.4. Computational time complexity of AL iterations for solving the Stokes equations: the linear solve scales linearly with the system size n , or the total number of faces in the mesh.

3.4.2 Computational time complexity

The Lie advection of the nematic field is explicit. The computation is also local and parallelizable, which results in minimal computational cost. The principal cost of the algorithm stems from the assembly of differential matrices and the subsequent linear solve associated with the AL iteration. The assembly of matrices, however, needs to be precomputed only once. Thus, the predominant computation load is attributed to the Poisson solve $A \mathbf{x} = \mathbf{b}$ for the Stokes equations in each AL iteration. In this study, A is a symmetric positive definite Laplacian matrix. Efficient solutions can be attained by standard sparse linear algebra libraries. For smaller system sizes, we employ the direct solve in Python’s `Scipy.sparse` library. As the system size increases, an iterative method such as conjugate gradient method becomes more pertinent. Empirically, as observed in Fig. 3.4, the computation time for the AL iteration exhibits linear scaling with the system size n , where n is the total number of faces in the mesh.

These tests were conducted on a Dell 8940 Desktop equipped with an Intel CPU i9-11900K operating at 3.50 GHz, 64 GB of RAM. In instances where a finer mesh is employed, comprising approximately 10,000 vertices—akin to the examples illustrated in the paper—the computation at each time step requires approximately 2 minutes.

3.4.3 Nematodynamics on arbitrary surfaces

The application of our method extends beyond idealized geometries. To demonstrate its generality on arbitrary geometries and topologies, we apply it to solve the full nematodynamics equations in biophysically relevant examples. We initialize the system with a smooth nematic field by calculating the eigenmodes of the nematic Laplacian $\Delta^L[\mathbf{q}] = \lambda[\mathbf{q}]$ corresponding to eigenvalues λ close to zero. These eigenmodes approximate minimizers of the Frank-Oseen energy. The numerical experiments encompass analytical shapes such as spheres and genus-1 tori, and organic shapes characterized by arbitrary curvature distributions. As previously mentioned in Secs. 3.2.2 and 3.3.6, the Stokes equations have Killing vector fields as their kernel. On spheres and tori where such a Killing vector field exists, we carry out projections to remove the Killing component and focus on a specialized solution with a minimal L_2 norm. We present each example under two distinct levels of activity, distinguished by disparate Péclet numbers ($Pe = 1$ and $Pe = 10^4$, both extensile). They showcase both regular solutions and solutions in the regime of active turbulence.

To characterize the nematodynamics, we focus on recording key metrics, including defect counts (of charge $\pm 1/2$), the charge-curvature energy $\|d\chi\|^2 := \|\Sigma_i 2\pi \mathcal{Z}_{p_i} \delta_{p_i} - K\|_{\Delta^{-1}}^2$ (except on the sphere where curvature is uniform), flow velocity $\|\mathbf{u}\|^2$, and enstrophy $\|\mathcal{K}\mathbf{u}\|^2$. Here, χ represents the *geometric potential* [88, 118], defined by $\Delta\chi = 2\pi \mathcal{Z}_{p_i} \delta_{p_i} - K$ based on defect locations p_i and charges $\mathcal{Z}_{p_i}$ (cf. Sec. 3.2.1), where δ_{p_i} denotes the Dirac delta function centered at p_i . The enstrophy quantifies viscous dissipation and serves to assess sharp features and chaotic behavior. For each case at each Péclet number, we include visual snapshots captured at the end of the simulation. Taking Fig. 3.5 as an example, the top snapshot displays the texture of the nematic field, with positive defects shown in red and negative defects in blue. The bottom snapshot illustrates the streamlines of the fluid flow field, color-mapped to represent velocity

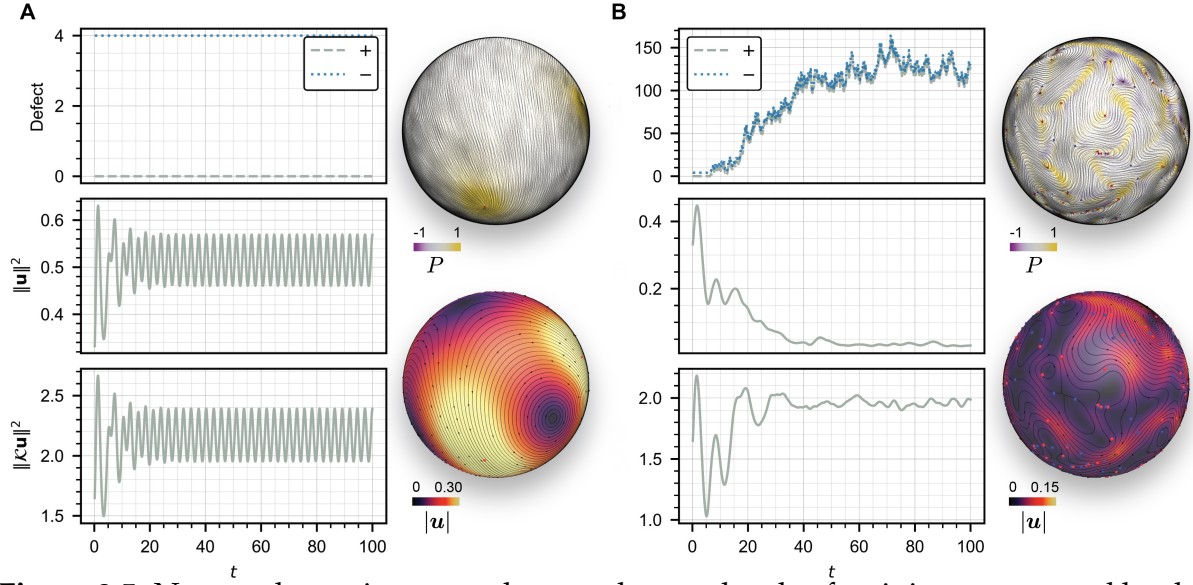


Figure 3.5. Nematodynamics on a sphere under two levels of activity, as captured by the Péclet number: A) low activity ($Pe = 1$), and B) high activity ($Pe = 10^4$). The animated time series simulations are available on YouTube: <https://youtu.be/gviiHbsWMsc>. From top to bottom, the time plots show: the total numbers of $+1/2$ and $-1/2$ topological defects of the nematic field, the L_2 norm squared $\|\mathbf{u}\|^2$ of the fluid velocity, and the enstropy $\|\mathcal{K}\mathbf{u}\|^2$ of the surface flow. Snapshots of the system are captured at the conclusion of each set of simulations ($t = 100$). The upper snapshots display the nematic field, color-mapped by the local power density exerted by the nematic onto the fluid, expressed as $P = -\langle \mathbf{u}, \text{div}^\nabla \mathbf{Q} \rangle$. The lower snapshots visualize the velocity field, color-mapped by its magnitude $|\mathbf{u}|$.

magnitude, $|\mathbf{u}|$. The colormap on the nematic field represents the local power density exerted by the active nematic on the fluid, given by $P = -\langle \mathbf{u}, \text{div}^\nabla \mathbf{Q} \rangle$. This expression captures the contribution of the active stress tensor to the fluid flow. Each positive half defect is accompanied by a self-propelling fluid jet and a vortex pair, and is surrounded by a region of positive power P . In contrast, negative half defects exhibit 3-atic symmetry and generate an index-2 flow field that is self-counteracting and remains static [44], contributing little energy to the fluid on large scales.

In Fig. 3.5A, we examine dynamics on a sphere at Péclet number $Pe = 1$, corresponding to a low activity level. Here, we observe a fixed number of four $+1/2$ defects exhibiting periodic orbiting motions—a phenomenon supported by previous empirical

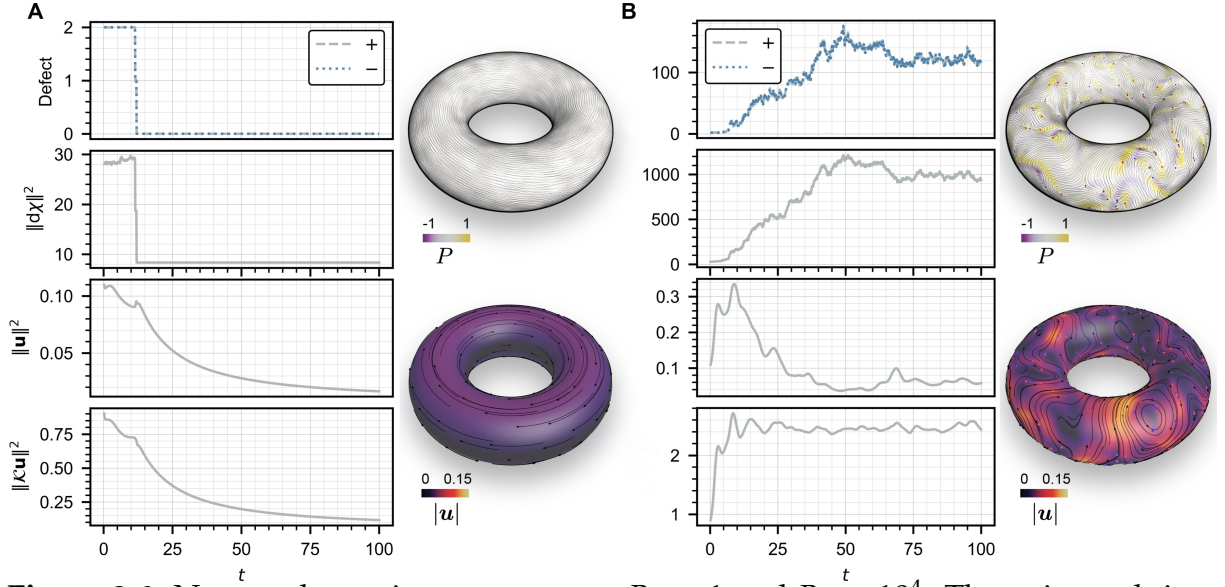


Figure 3.6. Nematodynamics on a torus at $Pe = 1$ and $Pe = 10^4$. The animated time series simulations are available on YouTube: <https://youtu.be/p4NLpHRTkPg>. The energy $\|d\chi\|^2$ measures the charge-curvature correlation. Refer to Fig. 3.5 for detailed descriptions of the other elements in the figure.

studies and simulations [57, 40]. In Fig. 3.5B, we increase the activity level to $Pe = 10^4$, leading to the emergence of active turbulence. Active turbulence is characterized by unsteady and chaotic dynamics driven by the spontaneous formation and annihilation of topological defect pairs. Demonstrated by Fig. 3.5B and consistent with the existing literature [112], the onset of turbulence is characterized by the appearance of large bending deformation walls that separate aligned nematic regions, the creation and annihilation of nematic defect pairs, and the separation of positive defects from negative defects due to self-propulsion of the positive. As the total number of topological defects increases, intricate flow patterns develop at smaller length scales. This observation is supported by the fluctuating but steady enstrophy and a rapid decrease in the overall velocity magnitude in the time series plots.

As our method is able to accommodate different topologies, we expand our exploration of nematodynamics to a genus one torus, characterized by a zero Euler characteristic. The vanishing Euler characteristic allows for a defect-free configuration

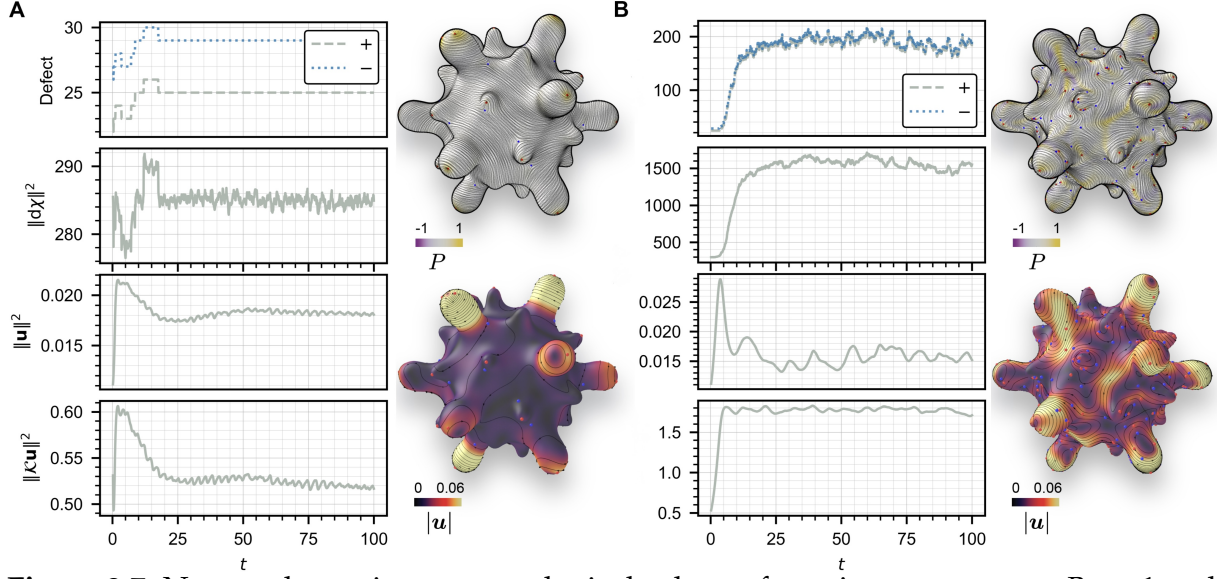


Figure 3.7. Nematodynamics on a topological sphere of varying curvature at $Pe = 1$ and $Pe = 10^4$. The animated time series simulations are available on Youtube: <https://youtu.be/hgsREDJwWZQ>. The energy $\|d\chi\|^2$ measures the charge-curvature correlation. Refer to Fig. 3.5 for detailed descriptions of the other elements in the figure.

at lower activity levels, as the nematic relaxation gradually merges two pairs of positive and negative defects (cf. Fig. 3.6A). A key observation is that the equilibrium defect-free nematic field on the torus displays rotational symmetry around its vertical axis, while maintaining a constant angle with the lines of toroidal coordinates. Consequently, the resulting Stokes flow also exhibits this rotational symmetry, with the toroidal midline acting as a shear layer. This flow pattern, in conjunction with diffusion, stabilizes the nematic field, allowing the system to reach a steady state. In Fig. 3.6B, when the Péclet number increases, the system transitions into a state of active turbulence through the injection of energy by topological defects, similar to what occurs on a sphere. Once again, we note that enstrophy reaches a plateau soon after the initial defect proliferation, while the velocity norm decreases as vortices form on increasingly smaller length scales. The charge-curvature energy rises with the number of defects.

In addition to standard analytical shapes, we applied our method to a topologically spherical membrane structure with an arbitrary curvature distribution (cf. Fig. 3.7A).

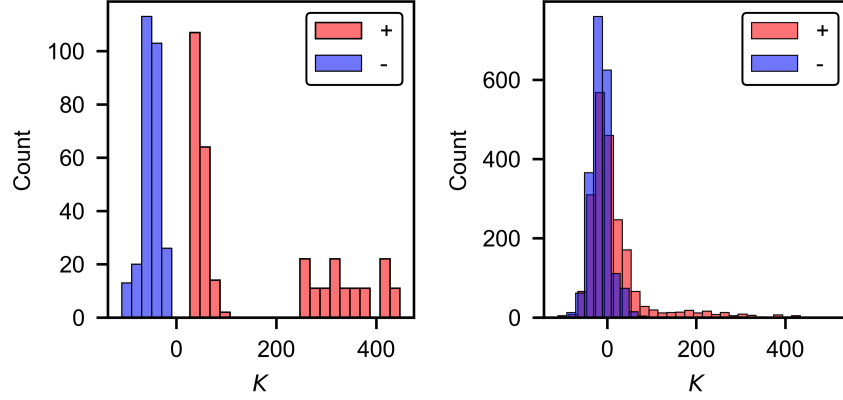


Figure 3.8. Frequency of defects as a function of Gaussian curvature for the spike-shaped geometry. Data are collected at steady state, sampled over $t = 50 : 5 : 100$. The left panel corresponds to $Pe = 1$, while the right panel corresponds to $Pe = 10^4$.

This geometry is generated through a simulation based on the Helfrich bending energy under the influence of an osmotic shock [130]. Even under low-activity conditions where nematic ordering relaxation prevails, a relatively large total number of defects persists. At the initial drop in $\|d\chi\|^2$ (cf. Fig. 3.7A and its animated movie), defects unbind and cluster in regions with like-sign curvature: positive defects gather at protrusions with positive curvature, while negative defects accumulate in negatively curved valleys (cf. Fig. 3.8). This behavior aligns with equilibrium theories of liquid crystals [118]. When two positive defects are present at a protrusion, they create strong circulating flows at the tips. These flows cause the defects to orbit periodically around the protrusions. This interaction at the protrusion is isolated from the rest of the domain, which remains almost motionless, thereby creating a limit cycle in the system. The distinct behaviors of active nematics in areas of positive and negative curvature suggests a possible detailed comparative study on active nematic systems in hyperbolic and elliptic geometries as a followup work. More broadly, the division of the flow field into areas of positive and negative curvature has implications for understanding how mixing and coherent structures are influenced by the shape of the domain. As Pe increases, nonlinear hydrodynamic effects disrupt the charge-curvature correlation (cf. Fig. 3.8),

consistent with observations in [88]. In addition, there is a notable difference in flow velocities when comparing conditions on a complex-shaped object to the sphere case. On a sphere, the interaction of topological defects creates complex flow patterns on small length scales, which tends to reduce the characteristic magnitude of the fluid velocity. However, as seen in Fig. 3.7B, for the complex shape, the magnitude of velocities remains relatively consistent whether in a steady state at low Pe or in active turbulence at high Pe . Active turbulence breaks down the stable coherent flow structures at low Pe , leading to a state with more efficient mixing by the flow. Specifically, this results in a decrease in velocity at the protrusions and an increase in velocity in the valleys of the manifold.

3.5 Conclusion

In this study, we formulate a minimal model for the dynamics of an active nematic fluid on a Riemannian 2-manifold. This model is cast based on a coordinate-free differential-geometric language and on the theory of the complex line bundle, which can naturally be generalized to curved spaces. We use the complex line to establish the 2D k -atic equivalence and model nematic advection through generalized Lie derivatives, laying the groundwork for extending the model to k -atic hydrodynamics. The Levi-Civita connection and its curvature are introduced within the framework of the complex line bundle. We show the coupling of the fluid's viscous stress with the curvature through the covariant Dolbeault $\bar{\partial}^\nabla$ operator. Numerically, our method evolves the active nematic system on a triangular mesh with arbitrary shape and topology. This is done by adhering to the discrete analogue of the continuous theory. We construct the Hermitian Bochner Laplacian matrix based on the variational principle and develop a generalized semi-Lagrangian scheme according to the Lie advection. This method maintains robustness and efficiency across various flow regimes, from low to high

activity levels.

In future work, we anticipate detailed studies that examine the relationship between the complex number representation and the well-established Q -tensor theory, including their generalizations to k -atic fields. While our theoretical framework is general, the current implementation is limited to closed manifolds; future work could incorporate boundary conditions. We acknowledge that our minimal Riemannian nematodynamics model does not account for reactive stresses due to nematic elasticity in advection [15], extrinsic geometry [87, 75], or the out-of-plane component of the nematic field [81]. These aspects will be addressed in future research. Depending on the specific application, it may also be important to consider fluid equations on evolving surfaces (cf. Chap. 1) [117, 2, 84, 85] and their coupling with bulk fluid [40, 67].

In summary, our integration of geometric language and complex manifold theory into the study of active nematics sets the stage for further theoretical and computational developments in this area. With further post-processing and analysis, we expect that the computational tool developed here will also be instrumental for specialists seeking deeper understanding of active nematics influenced by geometric aspects in specific systems. We hope that the incorporation of realistic geometric and topological features in the model can further narrow the gap between theoretical predictions and experimental findings.

Appendix

A Representation and navigation of discrete 2D manifolds

A closed, oriented triangular mesh M of arbitrary genus g consists of the sets of vertices, edges, and faces, denoted as $\{\mathfrak{V}, \mathfrak{E}, \mathfrak{F}\}$. Let $|\mathfrak{V}|$, $|\mathfrak{E}|$, and $|\mathfrak{F}|$ denote the numbers of vertices, edges, and faces, respectively. To indicate the local topology of

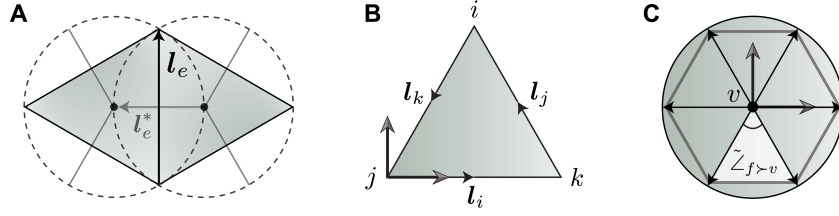


Figure 3.9. Schematics illustrating key components of a 2D triangular mesh: A) Voronoi dual mesh is constructed by circumcenters of the triangular mesh, defining the weight of each primal element. B) Tangent space associated with the faces of the mesh, denoted as $T_f M$. C) Vertex tangent space $T_v M$ via the geodesic polar map.

the mesh, we use the symbols $\prec$ and $\succ$, which signify the relation “incident to”. The orientation of the symbol indicates the dimensionality relationship. For example, for incident $v \in \mathfrak{V}$, $e \in \mathfrak{E}$, and $f \in \mathfrak{F}$, we write $v \prec e \prec f$. When using the summation sign, the iterating variable is placed first. For example, $\sum_{f \succ v}$ means summing over faces f that are adjacent to a given vertex v . The directed edge vector associated with the edge $e \in \mathfrak{E}$ in face $f \in \mathfrak{F}$, where $e \prec f$, is denoted as $l_{e \prec f}$. This vector is positively-oriented (i.e., counterclockwise) in face f . In particular, this edge has opposite orientations in the two faces that share it. Similarly we can also denote the edge vector associated with a vertex as $l_{v \prec f}$, which has the positive orientation of pointing outward from the vertex. Lastly, the angle of the interior corner of a triangle f at a vertex $v \prec f$ is denoted by $\angle_{v \prec f}$, $\angle_{f \succ v}$, $\angle_{e \prec f}$, or $\angle_{f \succ e}$, where e is the edge opposite to vertex v across triangle f .

Using the circumcenters of the mesh faces, we can construct a *Voronoi dual mesh*. In this dual mesh, each edge, known as the dual edge (illustrated by Fig. 3.9A), orthogonally bisects the corresponding primal edge in the primal mesh. Note that the dual edge is intrinsically straight, but it may be kinked when the mesh is embedded due to the dihedral angle formed by neighboring faces. We denote the directed dual edge as $l_{e \prec f}^* = w_e \mathbf{i} l_{e \prec f}$, where $w_e = |l_e|/|l_e^*|$ is the length ratio between the dual edge and the primal edge referred to as the *cotangent weight*², and $\mathbf{i}$ represents 90° counter-clockwise

²since $w_e = \frac{1}{2}(\cot(\angle_{e \prec f_1}) + \cot(\angle_{e \prec f_2}))$, where $f_1 \neq f_2 \succ e$ are the two incident faces of e .

rotation in face f . Analogous to the continuous notation that $C^\infty(M)$ denotes the space of continuous functions on M , we use $C_{\mathfrak{V}}(M) = \{h: \mathfrak{V} \rightarrow \mathbb{R}\}$ to denote the space of functions defined over vertices. The function evaluation at vertex $v \in \mathfrak{V}$ is denoted as h_v .

The notion of a tangent space to a mesh includes the tangent space $T_v M$ at a vertex v and the face tangent space $T_f M$ at a face f . A vector field defined on faces is a *face section* of the tangent bundle, $\Gamma_{\mathfrak{F}}(TM)$, and similarly, a vector field defined on vertices is a *vertex section* denoted as $\Gamma_{\mathfrak{V}}(TM)$. A vector basis section is represented by $\mathbf{1} \in \Gamma(TM)$, where the basis vector at a vertex v is denoted as $\mathbf{1}_v$, and at a face f as $\mathbf{1}_f$. Given a basis section, a vector field can be represented by a complex-valued function. For example, a face-based velocity field, $\mathbf{u} \in \Gamma_{\mathfrak{F}}(TM)$, can be represented as $\hat{\mathbf{u}}: \mathfrak{F} \rightarrow \mathbb{C}$. The same representation holds for a nematic field with its basis section denoted as $[\mathbf{1}] \in \Gamma(L)$. Specifically, on the face tangent space $T_f M$, we start by selecting an arbitrary positively-oriented normalized edge vector (e.g., the first edge e_1 in the data structure) as the basis, denoted as $\mathbf{1}_f = \mathbf{l}_{e_1 \prec f} / |\mathbf{l}_{e_1 \prec f}| \in T_f M$ and $[\mathbf{1}]_f = [\mathbf{1}_f] \in L_f$ (cf. Fig. 3.9B). Then, we can associate all vectors and nematic director in this tangent space with a complex number. In particular, we represent the phase angle of the oriented edge $\mathbf{l}_{e \prec f} = \hat{\mathbf{l}}_{e \prec f} \mathbf{1}_f$ as $\arg(\mathbf{l}_{e \prec f}) := \arg(\hat{\mathbf{l}}_{e \prec f})$. This phase angle is used to compute the discrete Levi-Civita connection in Sec. 3.3.1.

On a vertex tangent space $T_v M$, which is cone-shaped, we can construct an intrinsic representation referred to as the *geodesic polar map* (cf. Fig. 3.9C). To achieve this, we first need to flatten the cone. However, due to inherent curvature, a cone generally cannot be flattened perfectly due to angle defects or excesses. In other words, the sum of all corner angles of the faces incident to the vertex does not equal 2π (i.e., $\sum_{f \succ v} \angle_{f \succ v} \neq 2\pi$). To perform an intrinsic projection, we scale the corner angles by the ratio $r_v = (2\pi) / \sum_{f \succ v} \angle_{f \succ v}$. By applying this scaling factor to all interior angles at vertex v , we obtain a *geodesic angle* for each corner, denoted as $\tilde{\angle}_{f \succ v} = r_v \angle_{f \succ v}$. Similar

to the face tangent space, we choose an arbitrary positively-oriented edge vector $\mathbf{l}_{e_1 \prec v}$ at the vertex as the vector basis and its corresponding nematic basis, represented by $\mathbf{1}_v := \mathbf{l}_{e_1 \prec v} / |\mathbf{l}_{e_1 \prec v}| \in T_v M$ and $[\mathbf{1}]_v = [\mathbf{1}_v] \in L_v$. We accumulate the geodesic corner angles and obtain the phase angle $\arg(\mathbf{l}_{e \succ v}) = \arg(\hat{\mathbf{l}}_{e \succ v})$ for each edge e in this vertex polar plane. Consequently, the edge vector in $T_v M$ can be represented by a complex number $\mathbf{l}_{e \succ v} = \hat{\mathbf{l}}_{e \succ v} \mathbf{1}_v$.

B Derivation of $\bar{\partial}^\nabla \star \bar{\partial}^\nabla$ using complex differential form

In this section, we derive the viscosity Laplacian, $-2\bar{\partial}^\nabla \star \bar{\partial}^\nabla = \Delta + K : \Gamma(TM) \rightarrow \Gamma(TM)$, through the use of complex differential forms, assuming a basic knowledge of exterior calculus. Consider a Hermitian complex line bundle L over M with connection $\nabla : \Gamma(L) \rightarrow \Omega^1(M; L)$, where $\Omega^k(M; L)$ represents the space of L -valued k -forms. The curvature 2-form for the connection, $R^\nabla \in \Omega^2(M; \mathbb{C})$, is defined by

$$R^\nabla \llbracket v, w \rrbracket \psi := (d^\nabla \nabla \psi) \llbracket v, w \rrbracket = \nabla_v \nabla_w \psi - \nabla_w \nabla_v \psi - \nabla_{[v, w]} \psi. \quad (3.50)$$

Note that when $L = TM$ and ∇ is the Levi-Civita connection, R^∇ correlates directly with the Gaussian curvature K as $R^\nabla = -\mathbf{i}K \star 1$.

Note that for a 1-form $\alpha \in \Omega^1(M; L)$ on a 2-manifold, the relation $\alpha \llbracket \mathbf{i}v \rrbracket = -\star \alpha \llbracket v \rrbracket$ holds, where $\star$ denotes the Hodge star operator. The anti-holomorphic derivative, $\bar{\partial}^\nabla \psi$, $\psi \in \Gamma(L)$, induced by the connection ∇ , can be expressed as

$$\bar{\partial}^\nabla \psi := \frac{1}{2}(\nabla \psi + \mathbf{i} \nabla_{\mathbf{i}(\cdot)} \psi) = \frac{1}{2}(\nabla \psi - \mathbf{i} \star \nabla \psi). \quad (3.51)$$

The anti-holomorphic derivative allows us to construct the anti-holomorphic energy as

follows:

$$\begin{aligned}
2\langle \bar{\partial}^\nabla \psi, \bar{\partial}^\nabla \psi \rangle \star 1 &= \frac{1}{2} \langle \nabla \psi - \mathbf{i} \star \nabla \psi, \nabla \psi - \mathbf{i} \star \nabla \psi \rangle \star 1 \\
&= \frac{1}{2} (\langle \nabla \psi, \nabla \psi \rangle \star 1 + \langle \mathbf{i} \star \nabla \psi, \mathbf{i} \star \nabla \psi \rangle \star 1 - 2\langle \mathbf{i} \star \nabla \psi, \nabla \psi \rangle \star 1) \\
&= \langle \nabla \psi, \nabla \psi \rangle \star 1 + \langle \nabla \psi \wedge \mathbf{i} \nabla \psi \rangle \\
&= \langle \nabla \psi, \nabla \psi \rangle \star 1 + d\langle \psi \wedge \mathbf{i} \nabla \psi \rangle - \langle \psi \wedge \mathbf{i} d^\nabla \nabla \psi \rangle \\
&= \langle \nabla \psi, \nabla \psi \rangle \star 1 - \langle \psi, \star \mathbf{i} d^\nabla \nabla \psi \rangle = \langle \nabla \psi, \nabla \psi \rangle \star 1 - \langle \psi, \mathbf{i} \star R^\nabla \psi \rangle.
\end{aligned} \tag{3.52}$$

By taking a variation with respect to ψ on the final expression, we obtain the operator $-2\bar{\partial}^{\nabla*}\bar{\partial}^\nabla = \Delta + \mathbf{i} \star R^\nabla : \Gamma(L) \rightarrow \Gamma(L)$. When the complex line bundle is specialized to the tangent bundle, $L = TM$, the operator is the viscosity Laplacian, $2\bar{\partial}^{\nabla*}\bar{\partial}^\nabla = \Delta + K : \Gamma(TM) \rightarrow \Gamma(TM)$.

C Implementation software

SideFx Houdini is primarily known as a 3D animation and special effects software. However, it can be effectively utilized for custom algorithm implementation by leveraging its half-edge data structure of triangular meshes, as well as its VEX and Python programming capabilities. VEX is a C-like shader language that operates in a vectorized, multithreaded manner. The Python module enables access to the numerical linear algebra libraries Scipy. Additionally, the software provides visualization and basic geometric processing features such as mesh parametrization and interpolation to facilitate rapid prototyping.

Chapter 4

Conclusions and future directions

The dissertation author was the primary investigator and author of all work presented in this thesis.

This concluding chapter synthesizes the geometric variational framework developed throughout the thesis and outlines promising directions for future research. The preceding chapters have established a comprehensive approach to fluid mechanics on curved and evolving surfaces, demonstrating how differential geometry provides the unifying language for understanding complex flows across multiple regimes. Building on the foundational Stokes flows, the thesis progressively incorporated surface evolution, inertia, anisotropy and activity, revealing how geometric effects fundamentally alter fluid behavior in ways that cannot be captured by flat-space approximations.

The central contribution of this thesis is the development of a unified geometric framework that transcends traditional coordinate-based approaches. Rather than relying on local coordinates that obscure geometric structure and complicate implementation, the framework formulates the governing equations through differential geometry and variational principles. This yields coordinate-free descriptions of evolving surface Stokes flows, viscosity-induced geometric effects in vortex dynamics, and a complex-line formulation for active nematics. These formulations naturally lead to structure-preserving numerical methods. Collectively, these results demonstrate that geometry is

not merely a passive description of the fluid domain, but an active participant in the dynamics and physical principles governing fluid evolution.

Several physical insights also emerge from this geometric perspective. Surface curvature couples to fluid stresses in ways that drive spontaneous flows and organize topological defects. In nematic systems, the Gauss–Bonnet theorem imposes global topological constraints that lead to curvature-driven instabilities and defect migration. When the surface itself can deform, a feedback loop arises between shape evolution and fluid organization, enabling curvature-induced stabilization of active turbulence and sustaining ordered states at finite activity through shape–order coupling. In addition, Kutta conditions at sharp aerodynamic interfaces emerge naturally from frictionless free-slip boundaries, without requiring explicit boundary-layer transition models.

The geometric fluid framework developed here has broad practical applications. It advances our understanding of how curvature and activity organize flows in cellular membranes, developing tissues and cytoskeletal assemblies, enables the design of synthetic active systems that harness geometric effects for novel functionality, and provides new ways to control defect dynamics and phase transitions in curved active materials. Broadly, the computational methods developed here offer practical tools for modeling, simulation, and control, helping bridge theoretical predictions with increasingly advanced high-resolution experimental observations.

This work opens several promising directions for future research. One important avenue is integrating growth, fluidity changes, molecular-scale activity, and genetic regulation with tissue-scale geometry in developmental biology, linking biochemical signaling, cytoskeletal dynamics, and morphogenetic flows within a unified framework. Another is incorporating coupling between bulk flows and evolving surfaces, enabling stronger interactions between surrounding fluids and deforming interfaces. From a computational perspective, future work could integrate differentiable and optimal-control frameworks for parameter inference, inverse design, and data-driven control.

Extending the framework to handle topological changes, such as surface merging or splitting, would further broaden its applicability. Finally, developing higher-order numerical methods could improve both accuracy and convergence.

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